FREE VIBRATIONS ANALYSIS OF THIN PLATES BY THE BOUNDARY ELEMENT METHOD IN NON-SINGULAR APPROACH

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Abstract. A free vibration analysis of Kirchhoff plates is presented in the paper. Using proposed approach, there is no need to introduce Kirchhoff forces at the plate corner and equivalent shear forces at a plate boundary. Two unknown and independent variables are considered at the boundary element node. The Bettie theorem is used to derive the boundary integral equation. The collocation version of boundary element method with "constant" type of elements is presented. The source points are located slightly outside a plate boundary, hence the quasi-diagonal integrals of fundamental functions are non-singular.

Introduction

The Boundary Element Method (BEM) was created as a completely independent numerical tool to solve engineering problems [1, 2]. The BEM do not require the all domain discretization but only the boundary of a considered structure. This method reduces the computational dimension by one.

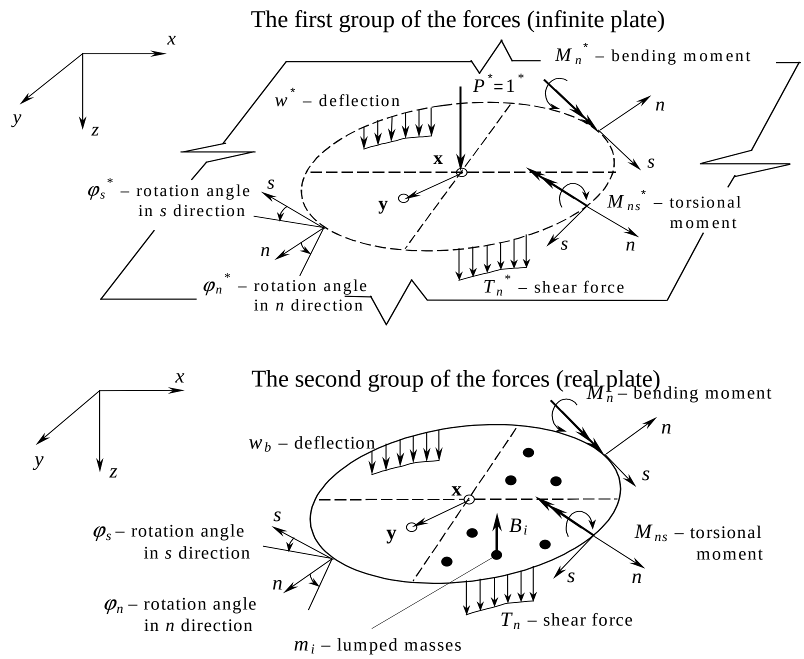
The Boundary Element Method is often used in the theory of both thin and thick plates and is particulary suitable to analyse the plates of arbitrary shapes and rested on internal supports. Analysis of plate bending using BEM was introduced by Bezine [3] and Stern [4] for Kirchhoff plate theory and by Vander Weeën [5] for the thick plate theory. Okupniak and Sygulski [6] used fundamental solution of Reissner plate proposed by Ganowicz [7]. Some authors present a modified approach of thin plate analysis. El-Zafrany, Debbih and Fadhil [8] assumed non-zero distribution of stress over the plate thickness. Hartley [9] proposed the BEM to solve similar problems. Guminiak [10, 11], Guminiak, and Sygulski [12] assumed a physical boundary condition also discussed in this paper.

Modelling of plate bending problem in free vibration analysis requires modification of governing boundary integral equation. Bèzine [3] proposed approach in which, the forces at the internal collocation points are treated as unknown variables. Katsikadelis *et al*. [13], Providakis and Toungelidis [14] and Shi [15] applied technique of Bezine to solve dynamic problems of thin plate. The second approach was proposed by Rashed [16] in application of a coupled BEM-flexibility force method in bending analysis of plates with internal supports.

Present paper includes a modified formulation for bending analysis of plates, in which three geometric and three static variables at the plate boundary are considered. In this formulation there is no need to introduce the equivalent shear forces at the boundary and concentrated forces at the plate corners. Similar to Hartley [9], the source points were located slightly outside a plate boundary, hence all of quasi-diagonal integrals are non-singular.

1. Integral formulation of thin plate bending in modified approach

On the plate boundary there are considered amplitudes of variables: the shear force T_n, bending moment M_n, torsional moment M_{ns} and deflection w, angle of rotation in normal direction φ_n and angle of rotation in tangent direction φ_s. Only two of them are independent. The boundary integral equation are derived using Bettie theorem. Two plates are considered: infinite plate, subjected unit concentrated loading and the real one (Fig. 1).



$\mathbf{x} = \mathbf{x}\,(x_1, x_2)$ - the source point
$\mathbf{y} = \mathbf{y}\,(x_1, x_2)$ - the field point

Fig. 1. Variables present in the boundary integral equation

A free vibration problem of thin plate is considered. In each internal collocation point associated with single lumped mass there are introduced displacement vector w_i, acceleration vector $\ddot{w}_i$ and inertial force B_i (Fig. 1)

$$w_i = W_i \sin \omega t \tag{1}$$

hence

$$\ddot{w}_i = -\omega^2 W_i \sin \omega t \tag{2}$$

Then, the inertial force amplitude is described

$$B_i = \omega^2 m_i W_i \tag{3}$$

where ω is the plate natural frequency. To derive integral equation the static fundamental solution is used. As a result the boundary integral equation are in the form:

$$\begin{aligned} &c(\mathbf{x}) \cdot w(\mathbf{x}) + \int_\Gamma \left[T_n^*(\mathbf{y},\mathbf{x}) \cdot w(\mathbf{y}) - M_n^*(\mathbf{y},\mathbf{x}) \cdot \varphi_n(\mathbf{y}) - M_{ns}^*(\mathbf{y},\mathbf{x}) \cdot \varphi_s(\mathbf{y})\right] d\Gamma(\mathbf{y}) = \\ &= \int_\Gamma \left[T_n(\mathbf{y}) \cdot w^*(\mathbf{y},\mathbf{x}) - M_n(\mathbf{y}) \cdot \varphi_n^*(\mathbf{y},\mathbf{x}) - M_{ns}(\mathbf{y}) \cdot \varphi_s^*(\mathbf{y},\mathbf{x})\right] \cdot d\Gamma(\mathbf{y}) + \\ &+ \sum_{i=1}^{I} B_i \cdot w^*(i,\mathbf{x}) \end{aligned} \tag{4}$$

where the fundamental solution of biharmonic equation

$$\nabla^4 w = \frac{1}{D} \cdot \bar{\delta}(\mathbf{y} - \mathbf{x}) \tag{5}$$

is given as a Green function

$$w^*(\mathbf{y},\mathbf{x}) = \frac{1}{D} \frac{r^2}{8\pi} \ln r \tag{6}$$

for a thin isotropic plate, $r = |\mathbf{y} - \mathbf{x}|$, $\bar{\delta}$ is Dirac delta and

$$D = \frac{E\, h_{\mathrm{p}}^3}{12\,(1 - \nu_{\mathrm{p}}^2)} \tag{7}$$

is a plate stiffness. The coefficient $c(\mathbf{x})$ is assumed as:

$c(\mathbf{x}) = 1$, when x is located inside the plate region,

$c(\mathbf{x}) = 0.5$, when x is located on the smooth boundary,

$c(\mathbf{x}) = 0$, when x is located outside the plate region.

The second equation can be derived by substituting of unit concentrated force $\mathrm{P}^{*}=1^{*}$ unit concentrated moment $M_{n}^{*}=1^{*}$. It is equivalent to differentiate the first boundary integral equation (8) on *n* direction in point **x** on a plate boundary

$$c(\mathbf{x})\cdot\varphi_{n}(\mathbf{x})+\int_{\Gamma}\left[\overline{T}_{n}^{*}(\mathbf{y},\mathbf{x})\cdot w(\mathbf{y})-\overline{M}_{n}^{*}(\mathbf{y},\mathbf{x})\cdot\varphi_{n}(\mathbf{y})-\overline{M}_{ns}^{*}(\mathbf{y},\mathbf{x})\cdot\varphi_{s}(\mathbf{y})\right]\cdot d\Gamma(\mathbf{y})=$$
$$=\int_{\Gamma}\left[T_{n}(\mathbf{y},\mathbf{x})\cdot\overline{w}^{*}(\mathbf{y},\mathbf{x})-M_{n}(\mathbf{y})\cdot\overline{\varphi}_{n}^{*}(\mathbf{y},\mathbf{x})-M_{ns}(\mathbf{y})\cdot\overline{\varphi}_{s}^{*}(\mathbf{y},\mathbf{x})\right]\cdot d\Gamma(\mathbf{y})+ \tag{8}$$
$$+\sum_{i=1}^{I}B_{i}\cdot\overline{w}^{*}(i,\mathbf{x})$$

where

$$\left\{\overline{T_{n}^{*}}(\mathbf{y},\mathbf{x}),\overline{M_{n}^{*}}(\mathbf{y},\mathbf{x}),\overline{M_{ns}^{*}}(\mathbf{y},\mathbf{x}),\overline{w^{*}}(\mathbf{y},\mathbf{x}),\overline{\varphi_{n}^{*}}(\mathbf{y},\mathbf{x}),\overline{\varphi_{s}^{*}}(\mathbf{y},\mathbf{x})\right\}=$$
$$=\frac{\partial}{\partial n(\mathbf{x})}\left\{T_{n}^{*}(\mathbf{y},\mathbf{x}),M_{n}^{*}(\mathbf{y},\mathbf{x}),M_{ns}^{*}(\mathbf{y},\mathbf{x}),w^{*}(\mathbf{y},\mathbf{x}),\varphi_{n}^{*}(\mathbf{y},\mathbf{x}),\varphi_{s}^{*}(\mathbf{y},\mathbf{x})\right\} \tag{9}$$

2. Boundary conditions

2.1. Clamped boundary

The boundary conditions are formulated as follows:

$$\begin{cases} w=0\\ \varphi_{n}=0\\ \varphi_{s}=0\\ M_{ns}=0\end{cases} \tag{10}$$

The unknown variables are: the bending moment M_n and the shear force T_n (Fig. 2).

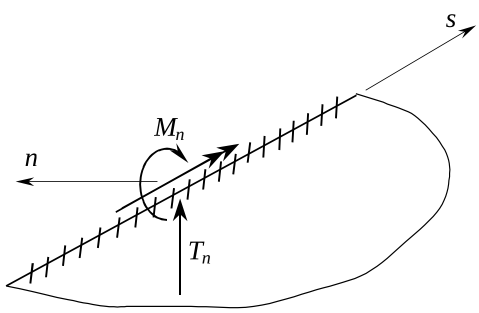


Fig. 2. Variables presented on the clamped edge

2.2. Simply-supported boundary

The boundary conditions are formulated as follows:

$$\begin{cases} w = 0 \\ \varphi_s = 0 \\ M_n = 0 \\ M_{ns} = 0 \end{cases} \tag{11}$$

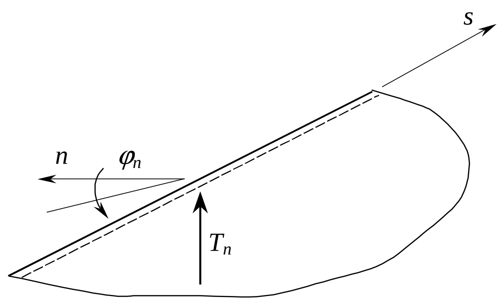


Fig. 3. Variables presented on the simply-supported edge

The unknown values are: the shear force T_n and the angle of rotation in direction *n*, φ_n (Fig. 3).

2.3. Free boundary

The boundary conditions are formulated as follows:

$$\begin{cases} T_n = 0 \\ M_n = 0 \\ M_{ns} = 0 \end{cases} \tag{12}$$

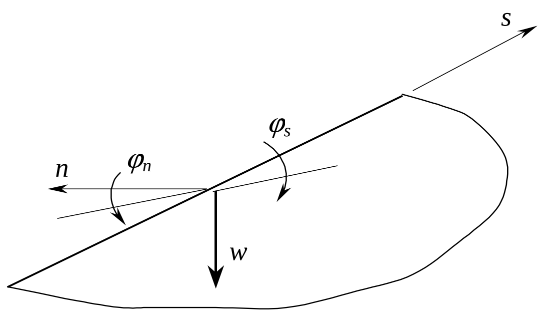


Fig. 4. Variables presented on the free edge

The unknown variables are: the deflection w and the angles of rotation φ_n, φ_s (Fig. 5). Because the relation between φ_s and w is known, $\varphi_s = \frac{\partial w}{\partial s}$, there are only two independent values: w and φ_n. After discretization of a plate boundary into constant elements having the same length, parameter $\frac{\partial w}{\partial s}(\mathbf{y})$ can be calculated approximately by constructing a differential expression using deflections of three neighbouring nodes (Fig. 5).

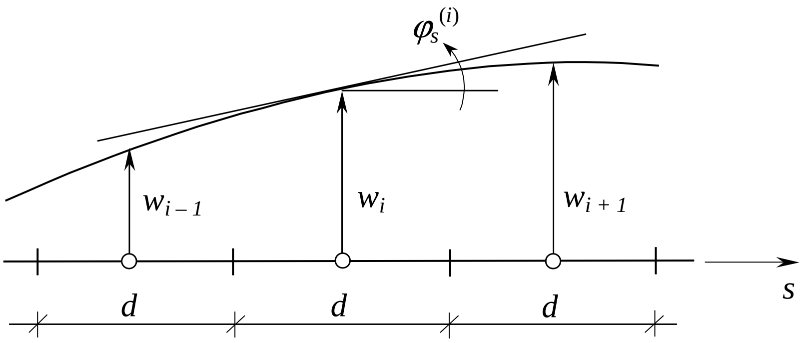


Fig. 5. Calculation of angle of rotation in tangent direction

$$\varphi_s^{(i)} = \frac{1}{2d}\left(w_{i+1} - w_{i-1}\right) \tag{13}$$

$$\varphi_s^{(i-1)} = \frac{1}{d}\left(-\frac{3}{2}w_{i-1} + 2w_i - \frac{1}{2}w_{i+1}\right) \tag{14}$$

$$\varphi_s^{(i+1)} = \frac{1}{d}\left(\frac{1}{2}w_{i-1} - 2w_i + \frac{3}{2}w_{i+1}\right) \tag{15}$$

The expressions (13) and (15) are needed for the nodes located on the left and right end of the free boundary.

3. Construction of set of algebraic equation

The set of algebraic equation can be written in the form:

$$\begin{bmatrix} \mathbf{G}_{\mathbf{XX}} & -\lambda\,\mathbf{E}_{\mathbf{Xw}} \\ \mathbf{G}_{\mathbf{wX}} & -\lambda\,\mathbf{E}_{\mathbf{ww}}\,\mathbf{M}_{\mathrm{p}} + \mathbf{I} \end{bmatrix} \begin{Bmatrix} \mathbf{X} \\ \mathbf{w} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix} \tag{16}$$

where

$$\mathbf{M}_{\mathrm{p}} = \mathrm{diag}(m_1, m_2, ..., m_N) \tag{17}$$

$\lambda = \omega^2$, $\mathbf{I}$ is the unit matrix and N is the number of lumped masses. The elements of characteristic matrix: $\mathbf{G}_{\mathbf{XX}}$ and $\mathbf{G}_{\mathbf{wX}}$ contain integrals of suitable fundamental functions depended from type of boundary (Fig. 6). These integrals are calculated in local coordinate system n_i, s_i and then transformed to coordinate system n_k, s_k. The quasi-diagonal elements of characteristic matrix were calculated analytically and rest of them numerically using 12-point Gauss quadrature.

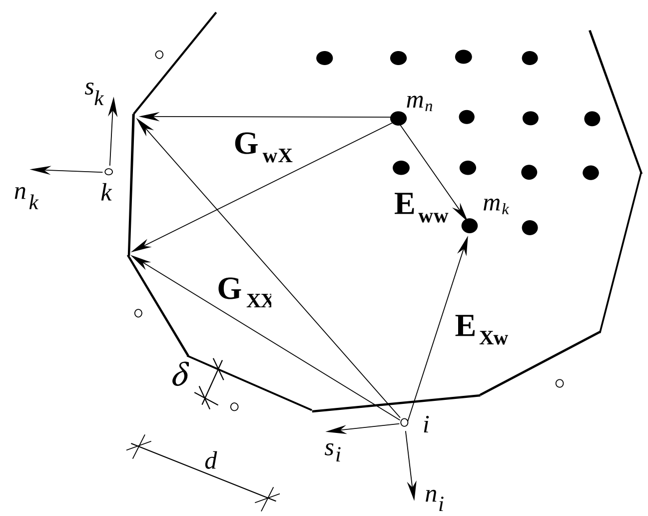


Fig. 6. Construction of set of algebraic equation

The second matrix equation in the set of equation (16) is obtained by construction the boundary integral equations for internal collocation points. Elimination of boundary variables $\mathbf{X}$ from matrix equation (16) leads to a standard eigenvalue problem:

$$\left\{\mathbf{A} - \tilde{\lambda} \cdot \mathbf{I}\right\}\mathbf{w} = \mathbf{0} \tag{18}$$

where $\tilde{\lambda} = 1/\omega^2$ and

$$\mathbf{A} = \left\{\mathbf{E}_{\mathbf{ww}}\,\mathbf{M} - \mathbf{G}_{\mathbf{wX}}\,[\mathbf{G}_{\mathbf{XX}}]^{-1}\,\mathbf{E}_{\mathbf{Xw}}\,\mathbf{M}\right\} \tag{19}$$

4. Fluid-plate interaction

A fluid is a source of additional inertia forces resulting from its mass and forces of radiation damping associated with energy dissipation. It is assumed that the plate is surrounded from all sides by the infinite fluid which is incompressible and inviscid. The velocity potential of the fluid for small disturbations is given in the form:

$$\varphi(\mathbf{x},t)=\tilde{\varphi}(\mathbf{x})\cdot e^{i\omega t} \tag{20}$$

where $\mathbf{x}=(x,y,z)$ and $\tilde{\varphi}(\mathbf{x})$ satisfies the Laplace equation:

$$\nabla^2\tilde{\varphi}(\mathbf{x})=0 \tag{21}$$

and ω is the circular frequency. The solution of equation (21) can be expressed in terms the double layer potential by the following boundary integral equation:

$$\tilde{\varphi}(P)=\int_S(\tilde{\varphi}_1(Q)-\tilde{\varphi}_2(Q))\frac{\partial\varphi^*(P,Q)}{\partial z_Q}dS_Q \tag{22}$$

where

$$\varphi^*(P,Q)=\frac{1}{4\pi}\cdot\frac{1}{r(P,Q)} \tag{23}$$

is the fundamental solution of the Laplace equation (21), $\tilde{\varphi}_1(Q)$ and $\tilde{\varphi}_2(Q)$ are the amplitudes of the velocity potential above and below the surface, $\tilde{\varphi}(P)$ is the amplitude of the velocity potential at any point of the space.

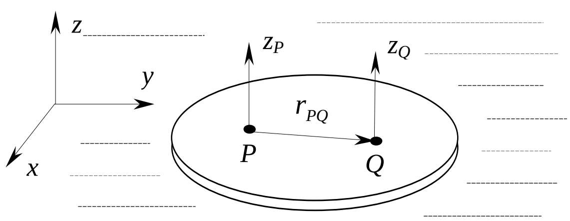


Fig. 7. Calculation of the fluid velocity potential

The hydrodynamic pressure acting on the plate surface:

$$p=-\rho_{\mathrm{f}}\frac{\partial\varphi(\mathbf{x},t)}{\partial t} \tag{24}$$

Assuming that $w(\mathbf{x},t)=\tilde{w}(\mathbf{x})e^{i\omega t}$ is the normal displacement of the plate structure, calculating the derivative of the fluid velocity potential and introducing the boundary condition of the Neumann's type, it is possible to obtain:

$$-\rho_{\mathrm{f}}\,\omega^2\tilde{w}(P)=\int_S \Delta p(Q)\frac{\partial^2\varphi^*(P,Q)}{\partial z_P\partial z_Q}dS_Q \tag{25}$$

where $\Delta p(Q)=\tilde{p}_2(Q)-\tilde{p}_1(Q)$ is the amplitude of the resultant hydrodynamic pressure at a point Q on the surface.

After discretization of the plate surface into sub-surfaces with area S_n, the amplitude of displacements in arbitrary point (x_m, y_m) can be joint with the hydrodynamic pressure amplitude:

$$\omega^2\tilde{w}(x_m,y_m)=-\frac{1}{4\pi\rho_{\mathrm{f}}}\sum_{n=1}^{N}\Delta p_n\int_{S_n}\frac{\partial^2}{\partial z_m^2}\left[\frac{1}{r}\right]\underset{z\to 0}{dS_n} \tag{26}$$

Equation (26) can be also written in a form:

$$-4\pi\rho_{\mathrm{f}}\,\omega^2\tilde{\mathbf{w}}=\mathbf{H}\tilde{\mathbf{p}} \tag{27}$$

where $\mathbf{H}$ is a ($N\times N$) - square matrix, in which all elements are defined by the equations (28)

$$H_{mn}=\int_{S_n}\frac{\partial^2}{\partial z_m^2}\left[\frac{1}{r}\right]\underset{z\to 0}{dS_n} \tag{28}$$

The hydrodynamic forces acting on the plate surface are:

$$\mathbf{P}=-\mathbf{M}_{\mathrm{f}}\,\omega^2\mathbf{w} \tag{29}$$

where

$$\mathbf{M}_{\mathrm{f}}=4\pi\rho_{\mathrm{f}}\mathbf{S}\mathbf{H}^{-1} \tag{30}$$

$P_n=\Delta p_n\,S_n$ and $\mathbf{S}=\mathrm{diag}(S_1 ... S_N)$ collects the areas of the individual sub-surfaces and N is the number of internal sub-surfaces. Internal sub-surfaces are treated as a boundary elements of the "constant" type for surrounding fluid.

Now, the set of algebraic equation describing vibration of a plate immersed in fluid has a form:

$$\begin{bmatrix}\mathbf{G}_{\mathbf{XX}} & -\lambda\,\mathbf{E}_{\mathbf{Xw}}\\ \mathbf{G}_{\mathbf{wX}} & -\lambda\,\mathbf{E}_{\mathbf{ww}}\tilde{\mathbf{M}}+\mathbf{I}\end{bmatrix}\begin{Bmatrix}\mathbf{X}\\ \mathbf{w}\end{Bmatrix}=\begin{Bmatrix}\mathbf{0}\\ \mathbf{0}\end{Bmatrix} \tag{31}$$

where

$$\tilde{\mathbf{M}}=\mathbf{M}_{\mathrm{p}}+\mathbf{M}_{\mathrm{f}}$$

5. Numerical examples

A rectangular and skew plates are considered. The results of calculation are verified using papers [17-19]. The set of boundary elements is regular. Each plate edge is divided by elements of the same length. The set of lumped masses is regular. The collocation points of boundary elements are located slightly outside the plate edge: $\varepsilon = \delta / d$.

The i^{th} natural frequency can be expressed as follow:

$$\omega_i = \frac{\mu_i}{l^2}\sqrt{\frac{D}{\rho_{\mathrm{p}} \cdot h_{\mathrm{p}}}} \tag{32}$$

where ρ_{p} is plate density and coefficients μ_i are presented in tables for every example.

5.1. A simply-supported square plate

Number of boundary elements: 120, number of lumped masses: 100.
The plate properties: E_{p} = 205 GPa, v_{p} = 0.3, h_{p} = 0.01 m, h = l = 2.0 m, $\rho_{\mathrm{p}} = 7850$ kg/m^3, $\varepsilon = \delta/d = 0.01$.

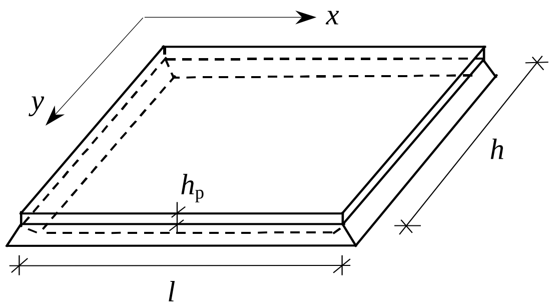


Fig. 8. A simply-supported plate

Table 1

Numerical results for the simply-supported plate

Frequency	Coefficients μ_i	
	Analytical solution [17]	Present solution **MEB**
1	19.73912	19.74041
2 and 3	49.34792	49.32904
4	78.95673	78.87008
5	98.96610	98.49901

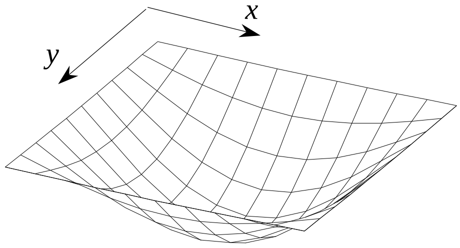


Fig. 9. The first mode

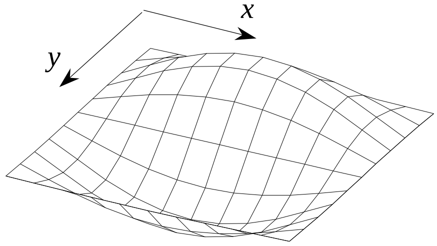


Fig. 10. The second mode

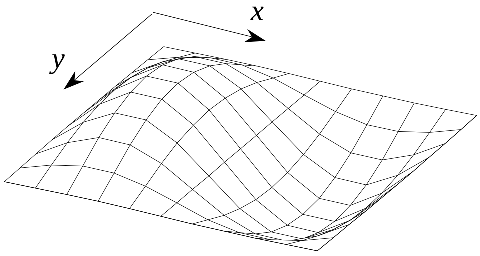


Fig. 11. The third mode

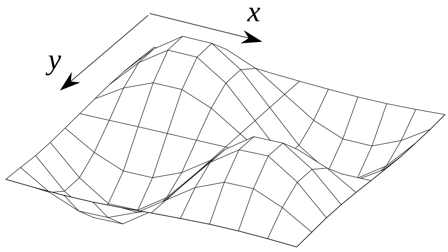


Fig. 12. The fourth mode

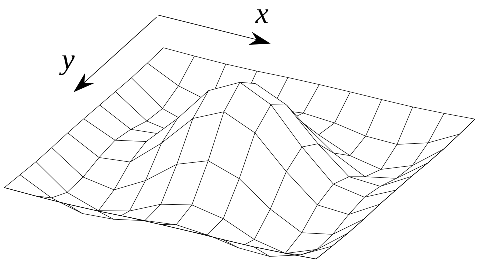


Fig. 13. The fifth mode

5.2. A clamped square plate

Number of boundary elements: 120, number of lumped masses: 100.
The plate properties: E_p = 205 GPa, ν_p = 0.3, h_p = 0.01 m, h = l = 2.0 m, $\rho_p = 7850$ kg/m^3, $\varepsilon = \delta / d = 0.01$.

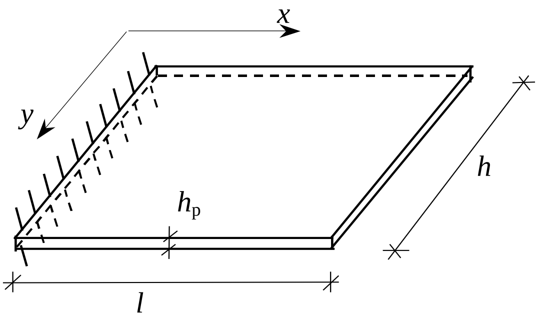


Fig. 14. A clamped plate

Table 2

Numerical results for the clamped plate

Frequency	Coefficients μ_i	
	MES solution [18]	Present solution **MEB**
1	3.49298	3.46216
2	8.54690	8.49957
3	21.44006	21.50939
4	27.45986	27.61505
5	31.17.008	31.10153

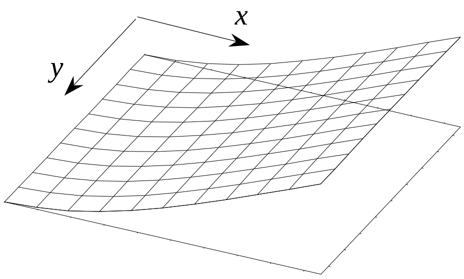


Fig. 15. The third mode

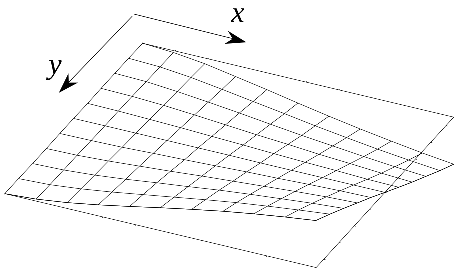


Fig. 16. The fourth mode

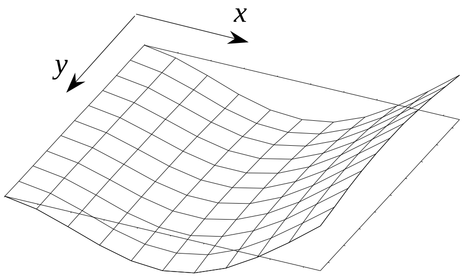


Fig. 17. The third mode

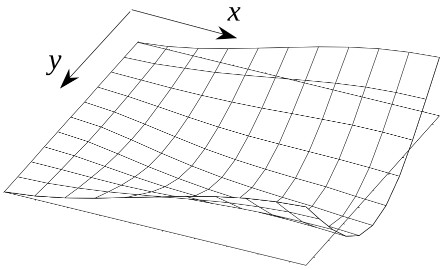


Fig. 18. The fourth mode

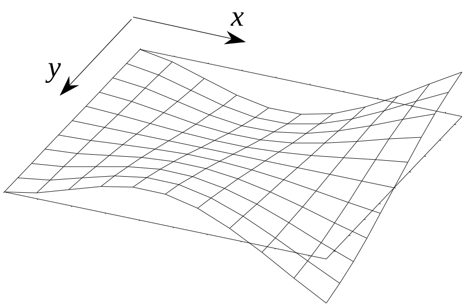


Fig. 19. The fourth mode

5.3. A clamped square plate immersed in fluid

Number of boundary elements: 120, number of lumped masses: 100.
The plate properties: E_p = 205 GPa, v_p = 0.3, h_p = 0.05 m, $h = l$ = 2.0 m, $\rho_p = 7850$ kg/m^3, $\rho_f = 1000$ kg/m^3, $\varepsilon = \delta/d = 0.02$.

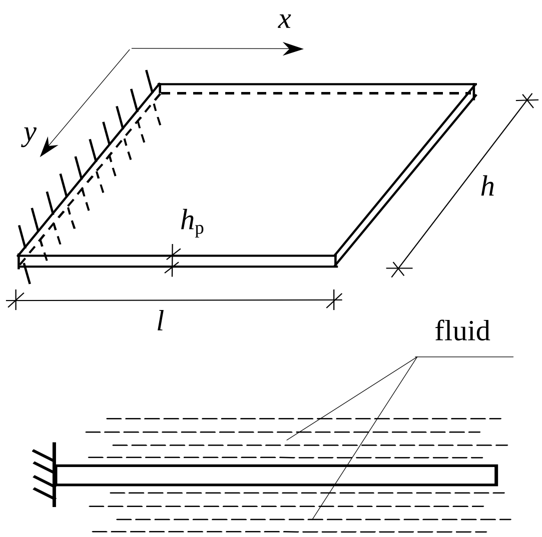


Fig. 20. A clamped square plate immersed in fluid

Table 3

Numerical results for the clamped plate immersed in fluid

Frequency	Coefficients μ_i	
	Analytical solution [19]	Present solution **MEB**
1	1.80602	1.94414
2	5.90702	5.45146
3	11.00598	13.27127
4	19.48201	18.62264
5	21.56800	20.90014

The lowest modes from first mode to fifth mode are similar to example 5.3.

5.4. A skew plate, simply-supported on two opposite edges

Number of boundary elements: 120, number of lumped masses: 100.
The plate properties: E_p = 205 GPa, v_p = 0.3, h_p = 0.02 m, $l_x = 1.5$ m, $l = l_y = 1.0$ m, $\rho_p = 7850$ kg/m^3, $\varepsilon = \delta/d = 0.02$.

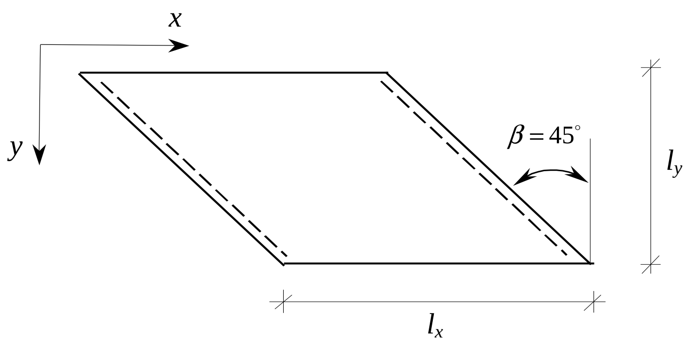


Fig. 21. A skew plate, simply-supported on two opposite edge

Table 4

Numerical results for the clamped plate

Frequency	Coefficients μ_i
	Present solution **MEB**
1	7.16064
2	9.06183
3	18.70131
4	27.35135
5	36.75122

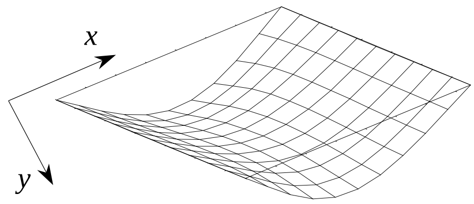


Fig. 22. The third mode

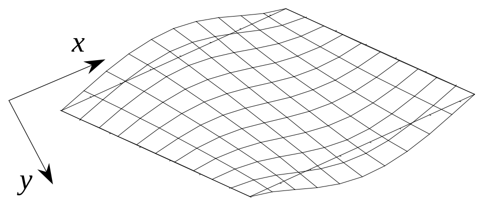


Fig. 23. The fourth mode

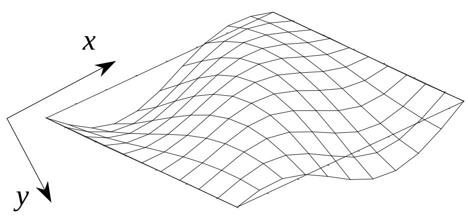


Fig. 24. The third mode

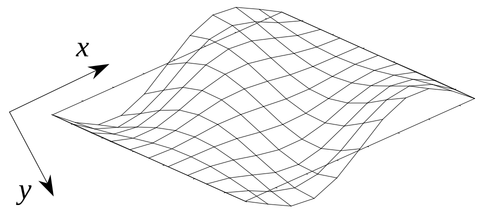


Fig. 25. The fourth mode

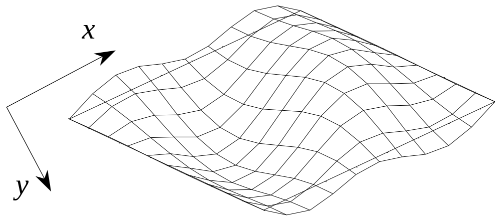


Fig. 26. The fifth mode

Conclusions

In this paper a linear theory of free vibration analysis of thin has been presented. The boundary element method with modified formulation of boundary conditions was used as a numerical tool. In this formulation, there is no need to introduce the Kirchhoff forces at a plate corners and the equivalent shear forces at a plate boundary. The collocation version of boundary element method with constant elements and non-singular calculations of integrals are employed. The source points of the boundary elements are located slightly outside a plate boundary, hence all of integrals of fundamental function are non-singular. The displayed boundary element results demonstrate the effectiveness and efficiency of the proposed method, which can be applied especially in skew plates bending analysis. Obtained BEM numerical results were compared with results taken from analytical and finite element way. This approach can be useful in engineering analysis of structures.

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