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# PSEUDO-RIEMANNIEN METRIC OF THE SIGNATURE (n,n) ASSOCIATED WITH A SYMPLECTIC FORM ON A 2n-DIMENSIONAL MANIFOLD

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**Abstract.** In the paper the pseudo-Riemannien structure of the signature (n, n) has been applied a connection of this symplectic structure.

Let M be a 2n-dimensional smooth differentiable manifold. By $F(M)$ we denote the ring of all smooth real functions on $M$ and by $\aleph(M)$ we denote the $F(M)$ - module of all smooth vector fields tangent to $M$. Next, let $TM$ be the tangent bundle of $M$ and $T_X M$ be the tangent space to $M$ at the point $x \in M$.

As it is well known, a skew-symmetric two-form $\omega$ on $M$ is called an almost - symplectic structure on M if the mapping

for $X \in \aleph(M)$, $X \mapsto l_X \omega = \omega(X,\cdot) \in \aleph^*(M)$

is an isomorphism, where $\aleph^*(M)$ is $F(M)$ - module dual to $\aleph(M)$. Moreover, if $d\omega = 0$, then $\omega$ is called a symplectic manifold. Now let $(M,\omega)$ be a symplectic manifold and $l: L \to M$ be an immersion. Then $L$ is called $\omega$-Lagrangian submanifold if

1. $\forall x \in L \quad l_x(T_X L) \subset T_X M$ is $\omega$ - isotropic subspace,
2. there exists subbudle $E \subset TL \oplus E$.

One can prove

Proposition 1. Let $(M,\omega)$ be a symplectic manifold and $L \subset M$ submanifold of $M$ if any only if

(1) dim L = $\frac{1}{2}$ dim $M$,

(2) $L$ is $\omega$ - isotropic submanifold, that means $\forall x \in L \quad l(T_X L) \quad \omega_X$ - isotropic subbudle of $T_X M$.

Evidently if $L$ is $\omega$-Lagrangian submanifold of $(M,\omega)$ then $TL$ is a $\omega$-Lagrangian of $TM$ submanifold that means $\omega(v,u)=0$ for any $vu \in TL$, as well as dim $TL$ $= \frac{1}{2}$ dim $TM$.

Let $(M,\omega)$ be a 2n-dimensional symplectic manifold and let

$$TM = W \oplus U \tag{1}$$

be decomposition into the direct sum of $\omega$-Lagrangian submanifolds $W$ and $U$.

Denote by

$$P: TM \to W \quad \text{and} \quad Q: TM \to U \tag{2}$$

the projections of $TM$ onto the submanifolds $W$ and $U$ parallel to the complementary submanifolds $U$ and $W$ respectively.

Obviously $P$ and $Q$ are smooth tensor fields of type (1) on $M$ satisfying the following conditions

$$P^2 = P \qquad Q^2 = Q \quad \text{and} \quad P + Q = id$$

In consequence the tensor field $t = P - Q$ on $M$ is an almost product structure on $M$. One can easily to prove the identity

$$\omega(X,Y) = \omega(P(X),Y) + \omega(X,P(Y)) \tag{3}$$

Now let us put

$$T(X,Y) = 2\omega(P(X),Y) \tag{4}$$

for any $X,Y \in \aleph(M)$. Hence by definitions $T$ is a smooth tensor field of type (2,0) on $M$ of rank $n$, that means rank $T_X = n$ for any $x \in M$.

From (3) and (4) it follows the identity

$$\omega(X,Y) = \frac{1}{2}\{T(X,Y) - T(Y,X)\} \tag{5}$$

with shows us that $\omega$ is equal to the skew-symmetric part $T$, that means $\omega = A(T)$.

Let now

$$g(X,Y) = \frac{1}{2}\{T(X,Y) + T(Y,X)\} \tag{6}$$

for arbitrary $X,Y \in \aleph(M)$. So, by definition $g = S(T)$ is a smooth symmetric tensor field on $M$, of type (2,0). We will show that g is a pseudo-Riemannien metric on $M$ of the signature $(n,n)$.

Really, let $x$ be an arbitrary point of $M$. Then there exists an open neighbourhood $V$ of x and smooth vector fields $X_1,...,X_n,Y_1,...,Y_n$ on $V$ such that

$$W_x = \langle X_1(x),...,X_n(x)\rangle \text{ and } U_x = \langle Y_1(x),...,Y_n(x)\rangle$$

Now it is easy to show that the matrix of the symplectic form $\omega(x)$ with respect to the basis $X_1(x),...,X_n(x)$, $Y_1(x),...,Y_n(x)$ has the form

$$\omega(x) \approx \begin{bmatrix} 0 & A \\ -A^t & 0 \end{bmatrix} \tag{7}$$

where $A = (\omega(X_i(x), Y_j(x)))$, $i, j = 1,2,\ldots.n$. Evidently $\det A \neq 0$ .

Using now (4) one can show that the matrix of the tensor $T_x$ with respect to the basis has the from

$$T_X \approx \begin{bmatrix} 0 & 2A \\ 0 & 0 \end{bmatrix} \tag{8}$$

Next from (6) and (8) it follows that the matrix of the form $g_x$ with respect to the considerable basis has the form

$$g_x \approx \begin{bmatrix} 0 & A \\ -A^t & 0 \end{bmatrix} \tag{9}$$

Evidently from (9) it follows that $g$ is of maximal rank.

Therefore $g$ is pseudo-Rimannien metric on $M$. Furthermore on can show that the signature of g is $(n, n)$.

Hence we have

Theorem 1. Let $(M,\omega)$ be a 2n-dimensional symplectic manifold. Then for every decomposition $TM = W \oplus U$ of the tangent bundle into the direct sum of $\omega$-Lagrangian submanifold there exists exactly one smoot tensor field $T$ on $M$, of type (2,0), defined by (4) such that

(1) $\omega = A(T)$

(2) $g = S(T)$ is a pseudo-Rimannien metric on $M$ of the signature $(n, n)$.

Evidently $T = \omega + g$ . Hence by definition (4) we get the identity

$$g(X,Y) = 2\omega(P(X),Y) - \omega(X,Y)$$

and consequently the identity

$$g(P(X),Y) = \omega(P(X),Y)$$

Using the last identity one can easily verify that the $\omega$-Lagrangian submanifold *W* and U in (1) are at the same time *g*-isotropic subbundles.

From Theorem (1) it follows

Corollary 1. Let $(M, \omega)$ be a 2n-dimensional symplectic manifold. Then there exists one-to-one correspondence between almost product structures *t* of rank *n* on *M*, on the hand, and the tensor fields *T* on *M* defined By (4) on the other hand.

## References

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