THE POLYNOMIAL INTERPOLATION FOR TECHNICAL EXPERIMENTS

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Abstract. In the article there are the interpolation of the empirical processes by polynomials of several variables.

Introduction

The interpolation formulas by polynomials of several variables are the unknown in the interpolation methods ([1]). In the empirical processes the measurement steps are constant. Using the Kronecker tensor product of matrices [2, 3] the interpolation formula was given in the theorem 1.

1. Kronecker's tensor product

Let $A_1^{(n_1)} \otimes A_2^{(n_2)} \otimes \ldots \otimes A_k^{(n_k)}$ be the Kronecker tensor product of matrices $A_1^{(n_1)}, A_2^{(n_2)}, \ldots, A_k^{(n_k)}$ of degrees respectively $n_1, n_2, \ldots, n_k$.

Proposition 1. *The determinant of the Kronecker tensor product is given by the formula*

$$\det\left(A_1^{(n_1)} \otimes A_2^{(n_2)} \otimes \ldots \otimes A_k^{(n_k)}\right) = \left(\det A_1^{(n_1)}\right)^{\hat{n}_1 n_2 \ldots n_k} \left(\det A_2^{(n_2)}\right)^{n_1 \hat{n}_2 \ldots n_k} \ldots \left(\det A_k^{(n_n)}\right)^{n_1 n_2 \ldots \hat{n}_k}$$

where the symbol $\wedge$ *omit above given numbers.*

Proof. This follows from the definition of the Kronecker tensor product by induction on degrees $n_1, n_2, \ldots, n_k$ [2, 3].

Let

$$X_i^{(p_i)} = \begin{bmatrix} 1 & X_{0i} & \cdots & X_{0i}^{p_i - 1_i} & X_{0i}^{p_{i_i}} \\ 1 & X_{0i} + \Delta_i & \cdots & (X_{0i} + \Delta_i)^{p_i - 1} & (X_{0i} + \Delta_i)^{p_i} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & X_{0i} + (p_i - 1)\Delta_i & \cdots & (X_{0i} + (p_i - 1)\Delta_i)^{p_i - 1} & (X_{0i} + (p_i - 1)\Delta_i)^{p_i} \\ 1 & X_{0i} + p_i \Delta_i & \cdots & (X_{0i} + p_i \Delta_i)^{p_i - 1} & (X_{0i} + p_i \Delta_i)^{p_i} \end{bmatrix}$$

be a matrix of degree $p_i + 1$ where $\Delta_i > 0$.

Proposition 2. *We have*

$$\det X_i^{(p_i)} = 2^{p_i - 1} 3^{p_i - 2} \ldots (p_i - 1)^2 \, p_i \, \Delta_i^{\frac{(p_i + 1) p_i}{2}}$$

Proof. It result immediately of the Vandermonde determinant [2, 3].
From both above propositions we obtain

Proposition 3. *The next tensor determinant is given by the fornula*

$$\det\left(X_1^{(p_1)} \otimes X_2^{(p_2)} \otimes \ldots \otimes X_k^{(p_k)}\right) = \\ = \prod_{i=1}^{k} \left(2^{p_i - 1} 3^{p_i - 2} \ldots (p_i - 1)^2 \, p_i \, \Delta_i^{\frac{(p_i + 1) p_i}{2}} \right)^{(p_1 + 1) \ldots (\hat{p}_i + 1) \ldots (p_k + 1)}$$

2. Polynomial interpolation

Consider the empirical process dependant on finite quantity of parameters with given initial conditions and measurement steps and know results. Thus let parameters $X_1, X_2, \ldots, X_k$ take initial positive values $X_{01}, X_{02}, \ldots, X_{0k}$ and the values $X_{01} + I_1\Delta_1, X_{02} + I_2\Delta_2, \ldots, X_{0k} + I_k\Delta_k$ with fixed positive steps $\Delta_1, \Delta_2, \ldots, \Delta_k$, where $0 \le I_1 \le p_1, 0 \le I_2 \le p_2, \ldots, 0 \le I_k \le p_k$. The results $W_{I_1 I_2 \ldots I_k}$ at all steps $X_{01} + I_1\Delta_1, X_{02} + I_2\Delta_2, \ldots, X_{0k} + I_k\Delta_k$ are know. So and the problem of polynomial interpolation follows to the question about to determine of coefficients of the polynomial

$$W(X_1, X_2, \ldots, X_k) = \sum_{\substack{0 \le i_1 \le p_1 \\ 0 \le i_2 \le p_2 \\ \vdots \\ 0 \le i_k \le p_k}} a_{i_1 i_2 \ldots i_k} X_1^{i_1} X_2^{i_2} \ldots X_k^{i_k}$$

for it

$$\sum_{\substack{0\le i_1\le p_1\\ 0\le i_2\le p_2\\ \vdots\\ 0\le i_k\le p_k}} a_{i_1 i_2 \dots i_k}\left(X_{01}+I_1\Delta_1\right)^{i_1}\left(X_{02}+I_2\Delta_2\right)^{i_2}\dots\left(X_{0k}+I_k\Delta_k\right)^{i_k}=W_{I_1 I_2 \dots I_k} \qquad (*)$$

where $0\le I_1\le p_1, 0\le I_2\le p_2,\dots, 0\le I_k\le p_k$.

Let $A_{i_1 i_2 \dots i_k}$ be the $i_1 i_2 \dots i_k$-th algebraic replacement of the matrix $X_1^{(p_1)}\otimes X_2^{(p_2)}\otimes\dots\otimes X_k^{(p_k)}$ of linear sysytems (*) by a column vector $W_{I_1 I_2 \dots I_k}$.

Theorem 1. *The linear systems of equations (*) has the unique solution and*

$$a_{i_1 i_2 \dots i_k}=\frac{\det A_{i_1 i_2 \dots i_k}}{\det\left(X_1^{(p_1)}\otimes X_2^{(p_2)}\otimes\dots\otimes X_k^{(p_k)}\right)}$$

for $0\le i_1\le p_1, 0\le i_2\le p_2,\dots,0\le i_k\le p_k$.

Proof. Take the order corresponding to the tensor product

$$a_{00\dots0}, a_{00\dots1},\dots, a_{00\dots p_k},\dots, a_{0p_2\dots p_k},\dots, a_{p_1 p_2\dots p_k}$$

Then the determinant of the system (1) is equal with it at the proposition 3 and the Cramer’s formulas are used.

3. Example

In the standard process of the nitriding under glow discharge ([4]) the parameters were as follow: temperature $T_0=793\,K$, $T_1=823\,K$, $T_2=853\,K$, pressure $p_0=150\,Pa$, $p_1=300\,Pa$ and time of the treatment $t_0=5\,h$, $t_1=10\,h$, $t_2=15\,h$. After nitriding surface layers were characterized by surface microhardness measurements $H_{T_i p_j t_k}$ for $0\le i\le 2$, $0\le j\le 1$ and $0\le k\le 2$.

So which the steps $\Delta T=30$ by temperature, $\Delta p=150$ by pressure and $\Delta t=5$ by time we obtain the next polynomial inerpolation formula for the surface microhardness

$$H(T,p,t)=\frac{1}{4\Delta T^{18}\Delta p^{9}\Delta t^{18}}\sum_{\substack{0\le i\le 2\\ 0\le j\le 1\\ 0\le k\le 2}}\left(\det A_{ijk}\right)T^i p^j t^k$$

where A_{ijk} denote the ijk-th algebraic replacement of the matrix

$$\begin{bmatrix} 1 & T_0 & T_0^2 \\ 1 & T_1 & T_1^2 \\ 1 & T_2 & T_2^2 \end{bmatrix} \otimes \begin{bmatrix} 1 & p_0 \\ 1 & p_1 \end{bmatrix} \otimes \begin{bmatrix} 1 & t_0 & t_0^2 \\ 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \end{bmatrix}$$

by a column vector $H_{T_i p_j t_k}$. (In this case the linear system (1) has the (18×18) type). After the normalization $\Delta\overline{T} = \dfrac{\Delta T}{10}$ and $\Delta\overline{p} = \dfrac{\Delta p}{10}$ we obtain

$$H(\overline{T}, \overline{p}, t) = \frac{1}{4 \cdot 3^{18} 19^9 5^{18}} \sum_{\substack{0 \le i \le 2 \\ 0 \le j \le 1 \\ 0 \le k \le 2}} \left(\det \overline{A}_{ijk}\right) \overline{T}^{\,i} \, \overline{p}^{\,j} t^k$$

where $\overline{A}_{ijk}$ denote the *ijk*-th algebraic replacement of the matrix

$$\begin{bmatrix} 1 & 52 & 52^2 \\ 1 & 55 & 55^2 \\ 1 & 58 & 58^2 \end{bmatrix} \otimes \begin{bmatrix} 1 & 15 \\ 1 & 30 \end{bmatrix} \otimes \begin{bmatrix} 1 & 5 & 5^2 \\ 1 & 10 & 10^2 \\ 1 & 15 & 15^2 \end{bmatrix}$$

by a column wector $\mathrm{H}_{\mathrm{T_i p_j t_k}}$.

References

[1] Kincaid D., Cheney W., Numerical Analysis. Mathematics of Scientific Computing, The University of Texas at Austin, 2002.

[2] Gantmacher F.R., The theory of matrices, Vol. 1, 2, Chelsea 1974.

[3] Lancaster P., Theory of Matrices, Acad. Press 1969.

[4] Grant Nr 3T08C06726, Modyfikacja powierzchni tytanu i jego stopów drogą azotowania w wyładowaniu jarzeniowym w celu poprawy własności eksploatacyjnych warstwy wierzchniej.