ON THE SOME SOLUTION OF THE L-Y EQUATION

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Abstract. In the article I have given the some solution of the Laplace-Young equation describing the shape of capillary surface.

Introduction

The Laplace-Young equation cannot be solved analytically in the global case ([1]). So it existe a some solution having continuous derivatives of all orders (propositon 2).

1. Solution of the problem

Consider a following basic form of the Laplace-Young equation [1, 2]

$$\left|\frac{d^2y}{dx^2}\right|\left(1+\left(\frac{dy}{dx}\right)^2\right)^{-\frac{3}{2}} - y^{-1}\left(1+\left(\frac{dy}{dx}\right)^2\right)^{-\frac{1}{2}} = p \qquad \text{(L-Y)}$$

where $p>0, y>0$ and $\frac{dy}{dx}>0$, $\frac{d^2y}{dx^2}<0$, with initial conditions $y(0)=1$, $\frac{dy}{dx}(0)=0$. Replacing $z=\frac{dy}{dx}$ the equation (L-Y) will becomes

$$\frac{z}{1+z^2}\frac{dz}{dy} = -p\sqrt{1+z^2} - \frac{1}{y} \qquad (1)$$

with the initial condition $z(1)=0$.

Proposition 1. *The solution of the equation (1) is given by*

$$z=\sqrt{y^{-2}(1+p\ln y)^{-2}-1}$$

for $e^{-\frac{1}{p}}<y\le 1$.
Proof. In fact, we have

$$1+z^2=\frac{1}{y^2(1+p\ln y)^2}$$

and

$$z\frac{dz}{dy}=-\frac{1+p\ln y+p}{y^3(1+p\ln y)^3}$$

so

$$\frac{z}{1+z^2}\frac{dz}{dy}=-\frac{1+p\ln y+p}{y(1+p\ln y)}=-\frac{1}{y}-p\frac{1}{y(1+p\ln y)}=-\frac{1}{y}-p\sqrt{1+z^2}$$

since $e^{-\frac{1}{p}}<y<1$.
Thus the solution of the equation (L-Y) yields to the solution of the equation

$$\frac{dy}{dx}=\sqrt{y^{-2}(1+p\ln y)^{-2}-1} \tag{2}$$

with the initial condition $y(0)=1$.

Proposition 2. *In the band* $e^{-\frac{1}{p}}<y<1$ *the equation (2) has the inegral*

$$\int\frac{dy}{\sqrt{y^{-2}(1+p\ln y)^{-2}-1}}=x+C \tag{3}$$

with the initial condition $y(0)=1$ *(after prolongation on the rigth).*
Replacing

$$z=y(1+p\ln y) \tag{4}$$

we obtain an integral solution

$$\int \frac{z}{\sqrt{1-z^2}} \frac{dz}{p+y^{-1}z} = x + C \tag{5}$$

where $e^{-\frac{1}{p}} < y < 1$ and $0 < z < 1$. The inverse function $y = y(z)$ given by the equation (4) defines correct the function

$$f(z) = \frac{1}{p + y^{-1}z}$$

where $0 < z \le 1$ and $e^{-\frac{1}{p}} < y \le 1$.

In the neighbourhood on the left of the point $z = 1$ ($y < 1$) we obtain the development of the function $f(z)$ in a series

$$f(z) = \frac{1}{p+1} - \frac{p}{(p+1)^3}(z-1) + \ldots$$

Thus the inegral (5) yields to the integral

$$\int \frac{z}{\sqrt{1-z^2}} \left(\frac{1}{p+1} - \frac{p}{(p+1)^3}(z-1) + \ldots \right) dz = x + C$$

with the initial condition $z(1) = 1$.

In the neighbourhood on the rigth of the point $z = 0$ ($y > e^{-\frac{1}{p}}$) we obtain the development of the function $f(z)$ in a series

$$f(z) = \frac{1}{p} - \frac{e^{\frac{1}{p}}}{p^2} z + \ldots$$

In this case the inegral (5) yields so to the integral

$$\int \frac{z}{\sqrt{1-z^2}} \left(\frac{1}{p} - \frac{e^{\frac{1}{p}}}{p^2} z + \ldots \right) dz = x + C$$

with the initial condition $z\left(e^{-\frac{1}{p}}\right) = 0$.

Remark. The solution of the equation (2) in the band $0 < y < e^{-\frac{1}{p}}$ is a subject of my next researchs [3].

References

[1] Han G., Rock stability under different fluid flow conditions, PhD thesis, Waterloo University, Ontario 2003.

[2] Błaszczuk J., Biernat G., Domański Z., Toroidal approximation for capillary bridges between ellipsoidal grains, Scientific Research of the Institute of Mathematics and Computer Science, Czestochowa University of Technology 2005, 1(4), 13-17.

[3] Biernat G., On the some solution of the L-Y equation. II. (to appear).