Abstract. In the article I have given the some solution of the Laplace-Young equation describing the shape of capillary surface.

Introduction

The Laplace-Young equation cannot be solved analytically in the global case ([1]). So it existe a some solution having continuous derivatives of all orders (proposition 2).

1. Solution of the problem

Consider a following basic form of the Laplace-Young equation [1, 2]

$$\frac{d^2 y}{dx^2} \left( 1 + \left( \frac{dy}{dx} \right)^2 \right)^{\frac{3}{2}} - y^{-1} \left( 1 + \left( \frac{dy}{dx} \right)^2 \right)^{\frac{1}{2}} = p$$

(L-Y)

where $p > 0$, $y > 0$ and $\frac{dy}{dx} > 0$, $\frac{d^2 y}{dx^2} < 0$, with initial conditions $y(0) = 1$, $\frac{dy}{dx}(0) = 0$. Replacing $z = \frac{dy}{dx}$ the equation (L-Y) will becomes

$$\frac{z}{1 + z^2} \frac{dz}{dy} = -p \sqrt{1 + z^2} - \frac{1}{y}$$

(1)

with the initial condition $z(1) = 0$. 

Please cite this article as:
The website: http://www.amcm.pcz.pl/
**Proposition 1.** The solution of the equation (1) is given by

\[ z = \sqrt{y^{-2}(1 + p \ln y)^2} - 1 \]

for \( e^{-\frac{1}{p}} < y \leq 1 \).

**Proof.** In fact, we have

\[ 1 + z^2 = \frac{1}{y^2(1 + p \ln y)^2} \]

and

\[ z \frac{dz}{dy} = -1 + p \ln y + p \]

\[ \frac{dy}{y^2(1 + p \ln y)^2} \]

so

\[ \frac{z}{1 + z^2} \frac{dz}{dy} = -\frac{1 + p \ln y + p}{y(1 + p \ln y)} = -\frac{1}{y} - \frac{1}{y(1 + p \ln y)} = -\frac{1}{y} - p\sqrt{1 + z^2} \]

since \( e^{-\frac{1}{p}} < y < 1 \).

Thus the solution of the equation (L-Y) yields to the solution of the equation

\[ \frac{dy}{dx} = \sqrt{y^{-2}(1 + p \ln y)^2} - 1 \]

(2)

with the initial condition \( y(0) = 1 \).

**Proposition 2.** In the band \( e^{-\frac{1}{p}} < y < 1 \) the equation (2) has the integral

\[ \int \frac{dy}{\sqrt{y^{-2}(1 + p \ln y)^2} - 1} = x + C \]

(3)

with the initial condition \( y(0) = 1 \) (after prolongation on the right).

Replacing

\[ z = y(1 + p \ln y) \]

(4)
we obtain an integral solution
\[
\int \frac{z}{\sqrt{1-z^2}} \frac{dz}{p + y^{-1}z} = x + C
\] (5)

where \( e^{\frac{-1}{p}} < y < 1 \) and \( 0 < z < 1 \). The inverse function \( y = y(z) \) given by the equation (4) defines correct the function
\[
f(z) = \frac{1}{p + y^{-1}z}
\]

where \( 0 < z \leq 1 \) and \( e^{\frac{-1}{p}} < y \leq 1 \).

In the neighbourhood on the left of the point \( z = 1 \ (y < 1) \) we obtain the development of the function \( f(z) \) in a series
\[
f(z) = \frac{1}{p+1} - \frac{p}{(p+1)^2} (z-1) + ...
\]

Thus the integral (5) yields to the integral
\[
\int \frac{z}{\sqrt{1-z^2}} \left( \frac{1}{p+1} - \frac{p}{(p+1)^2} (z-1) + ...ight) dz = x + C
\]

with the initial condition \( z(1) = 1 \).

In the neighbourhood on the right of the point \( z = 0 \ (y > e^{\frac{-1}{p}}) \) we obtain the development of the function \( f(z) \) in a series
\[
f(z) = \frac{1}{p} - \frac{e^p}{p^2} z + ...
\]

In this case the integral (5) yields so to the integral
\[
\int \frac{z}{\sqrt{1-z^2}} \left( \frac{1}{p} - \frac{e^p}{p^2} z + ... \right) dz = x + C
\]

with the initial condition \( z \left( e^{-\frac{1}{p}} \right) = 0 \).

Remark. The solution of the equation (2) in the band \( 0 < y < e^{-\frac{1}{p}} \) is a subject of my next researchs [3].

References