IDENTIFICATION OF SOURCE FUNCTION USING THE NUMERICAL METHODS

***Andrzej Metelski*[1], *Sylwia Lara*[2], *Edyta Pawlak*[2]**

[1] *Technological University of Opole, Poland*
[2] *Institute of Mathematics and Computer Science, Czestochowa University of Technology, Poland*

Abstract. In the paper the inverse heat conduction problem is discussed. The energy equation contains the term determining the capacity of volumetric internal heat sources $q(x)$ (x is a geometrical coordinate) and this parameter is identified. The solution of inverse problem bases on the least square criterion in which the sensitivity coefficients are introduced. In the final part of paper the examples concerning 1D and 2D problems are shown.

1. Governing equation

The heat conduction process proceeding in the solid body is described by the following energy equation [1]

$$c(T)\frac{\partial T(x,t)}{\partial t} = \nabla\left[\lambda(T)\,\nabla T(x,t)\right] + q(x) \tag{1}$$

where $c(T)$ is a volumetric specific heat, $\lambda(T)$ is a thermal conductivity, $q(x)$ is a capacity of internal heat sources, T, x, t denote the temperature, geometrical co-ordinates and time.
The equation (1) is supplemented by the boundary condition in a general form

$$\Phi\left[T(x,t), \frac{\partial T(x,t)}{\partial n}\right] = 0 \tag{2}$$

and an initial one

$$t = 0: \qquad T(x,0) = T_0(x) \tag{3}$$

It is assumed that the function $q(x)$ is dependent on geometrical coordinates and the parameters appearing in $q(x)$ will be identified.

2. Formulation of inverse problem

At first the 1D task is discussed. Assuming the constant values of c and λ the equation (1) takes a form

$$c\,\frac{\partial T(x,t)}{\partial t}=\lambda\,\frac{\partial^2 T(x,t)}{\partial x^2}+q(x) \tag{4}$$

Let us assume that $q(x)$ is given by formula

$$q(x)=p_1+p_2x+p_3x^2 \tag{5}$$

and the parameters p_1, p_2, p_3 are unknown [2-4].
In order to solve the identification problem the additional information concerning the temperature field in domain considered is necessary. So, we assume the knowledge of cooling (heating) curves at selected set of points x_i, i = 1,2,..., M, from domain analyzed. The cooling (heating) curves are given in discrete manner, namely

$$T_{d\,i}^{\,f}=T(x_i,t^f),\qquad i=1,2,\ldots,M,\qquad f=1,2,\ldots,F \tag{6}$$

where t^0, t^1, t^2,..., t^F are the successive levels of time.
The method proposed requires the constructions of sensitivity models of problem discussed with respect to unknown parameters p_1, p_2, p_3. So, the basic equation and conditions (2), (3) should be differentiated with respect to p_w, w = 1, 2, 3.
The sensitivity model with respect to p_w is the following

$$\begin{cases} x\in\Omega: & c\,\dfrac{\partial U_w(x,t)}{\partial t}=\lambda\,\dfrac{\partial^2 U_w(x,t)}{\partial x^2}+x^{w-1}\\ x\in\Gamma: & \Phi\left[U_w(x,t),\,\dfrac{\partial U_w(x,t)}{\partial n}\right]=0\\ t=0: & U_w(x,0)=0 \end{cases} \tag{7}$$

where $U_w(x,t)=\partial T(x,t)/\partial p_w$.
The least square criterion is introduced

$$S=\sum_{i=1}^{M}\sum_{f=1}^{F}\left(T_i^f-T_{d\,i}^{\,f}\right)^2\rightarrow \mathrm{MIN} \tag{8}$$

where T_i^f are the temperatures corresponding to arbitrary assumed values of p_w, w = 1,2,3.

The necessary condition of functional (8) min leads to the following system of equations

$$\frac{\partial S}{\partial p_e} = 2\sum_{i=1}^{M}\sum_{f=1}^{F}\left(T_i^{\,f} - T_{d\,i}^{\,f}\right)\frac{\partial T_i^{\,f}}{\partial p_e}\bigg|_{p_e = p_e^k} = 0 \tag{9}$$

Function $T = T(x,t)$ is expanded into Taylor series

$$T_i^{\,f} = \left(T_i^{\,f}\right)^k + \sum_{w=1}^{3}\frac{\partial T_i^{\,f}}{\partial p_w}\bigg|_{p_w = p_w^k}\left(p_w^{k+1} - p_w^k\right) \tag{10}$$

where $p_w^k,\ w = 1, 2, 3$ are the arbitrary but known values of parameters while k is iteration number.
Introducing (10) to (9) one obtains

$$\sum_{i=1}^{M}\sum_{f=1}^{F}\left[\left(T_i^{\,f}\right)^k - T_{d\,i}^{\,f}\right]U_{ie}^{\,f} + \sum_{i=1}^{M}\sum_{f=1}^{F}\sum_{w=1}^{3}U_{iw}^{\,f}U_{ie}^{\,f}\left(p_w^{k+1} - p_w^k\right) = 0 \tag{11}$$

Equation (11) can be written in a matrix form

$$\mathbf{A}\,\mathbf{p}^{k+1} = \mathbf{A}\,\mathbf{p}^k + \mathbf{B}^k \tag{12}$$

where

$$\mathbf{A} = \begin{bmatrix} \sum_{i=1}^{M}\sum_{f=1}^{F}\left(U_{i1}^{\,f}\right)^2 & \sum_{i=1}^{M}\sum_{f=1}^{F}U_{i1}^{\,f}U_{i2}^{\,f} & \sum_{i=1}^{M}\sum_{f=1}^{F}U_{i1}^{\,f}U_{i3}^{\,f} \\ \sum_{i=1}^{M}\sum_{f=1}^{F}U_{i2}^{\,f}U_{i1}^{\,f} & \sum_{i=1}^{M}\sum_{f=1}^{F}\left(U_{i2}^{\,f}\right)^2 & \sum_{i=1}^{M}\sum_{f=1}^{F}U_{i2}^{\,f}U_{i3}^{\,f} \\ \sum_{i=1}^{M}\sum_{f=1}^{F}U_{i3}^{\,f}U_{i1}^{\,f} & \sum_{i=1}^{M}\sum_{f=1}^{F}U_{i3}^{\,f}U_{i2}^{\,f} & \sum_{i=1}^{M}\sum_{f=1}^{F}\left(U_{i3}^{\,f}\right)^2 \end{bmatrix} \tag{13}$$

$$\mathbf{B}^k = \begin{bmatrix} \sum_{i=1}^{M}\sum_{f=1}^{F}\left[T_{d\,i}^{\,f} - \left(T_i^{\,f}\right)^k\right]U_{i1}^{\,f} \\ \sum_{i=1}^{M}\sum_{f=1}^{F}\left[T_{d\,i}^{\,f} - \left(T_i^{\,f}\right)^k\right]U_{i2}^{\,f} \\ \sum_{i=1}^{M}\sum_{f=1}^{F}\left[T_{d\,i}^{\,f} - \left(T_i^{\,f}\right)^k\right]U_{i3}^{\,f} \end{bmatrix} \tag{14}$$

$$\mathbf{p}^{k} = \begin{bmatrix} p_1^{k} \\ p_2^{k} \\ p_3^{k} \end{bmatrix}, \qquad \mathbf{p}^{k+1} = \begin{bmatrix} p_1^{k+1} \\ p_2^{k+1} \\ p_3^{k+1} \end{bmatrix} \tag{15}$$

Equation (12) allows to determine vector $\mathbf{p}^{k+1}$ on the basis of the knowledge of $\mathbf{p}^{k}$. The iterative process starts from arbitrary values of $p_1^0,\ p_2^0,\ p_3^0$ and the sequence $\{p_1\}$, $\{p_2\}$, $\{p_3\}$ (if the process is convergent) tend to values p_1, p_2, p_3.
As an example of 2D tasks parameters of linear function $q(x,\ y) = p_1 + p_2 x + p_3 y$ has been considered [5].
The algorithm of inverse problem solution is very similar to algorithm presented previously.

3. Example of computation

The plate $L = 0.01$ m for which $\lambda = 1.1$ W/mK, $c = 2.874$ MJ/m^{3}K and initial temperature T_0 = 1000$^{\text{o}}$C has been considered. For $x = 0$ the non-flux condition while for x = L the Robin one have been assumed ($\alpha = 60$ W/m^{2}K, T_a = 100$^{\text{o}}$C). Both the basic problem and the additional ones (resulting from sensitivity models) have been sold using FDM. The basic solution has been obtained under the assumption that $q(x) = -50000x^2 + 10000$ and the parameters appearing in this formula have been identified on the inverse problem solution. In Figure 1 the obtained temperature is shown.
The positions of sensor (control points) correspond to x_1 = 0.002 m, x_2 = 0.02 m, x_3 = 0.05 m, x_4 = 0.08 m, x_5 = 0.098 m. The cooling curves resulting from Figure 1 have been disturbed in random way in order to simulate the real measurements. After 10 simulations the mean values of unknown parameters are equal to p_1 = 9999.08, p_2 = 48.1, p_3 = −50463.5. One can see that the iteration is convergent and the identified parameters are close to the real values of $p_w^k,\ w = 1, 2, 3$. In Figure 5 the real and identified course of source function is shown.
In Figure 2-4 the distributions of sensitivity functions are marked.

The example illustrating 2D problem concerned the domain shown in Figure 6. Thermophysical parameters of material have been assumed the same as previously. Along the internal boundary the temperature T_0 = 1000$^{\text{o}}$C while along the external surface the Robin condition have been taken into account. The identified source function is of the form $q(x,y)$ = 9950 + 1000x − 1000y. In Figure 1 the temperature field after 1000 s is also shown. After 10 simulation the mean value of source function parameters have been found close to the real values, in particular $p_1 = 9948.9$, $p_2 = 1009.3$, $p_3 = -992.7$.

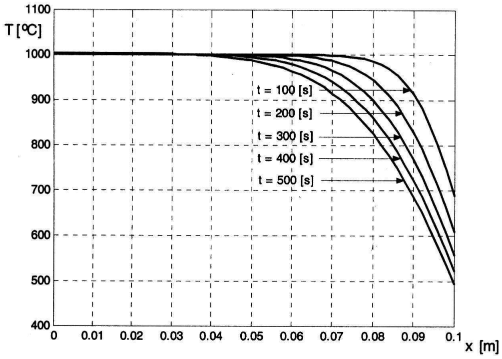


Fig. 1. Temperature distribution

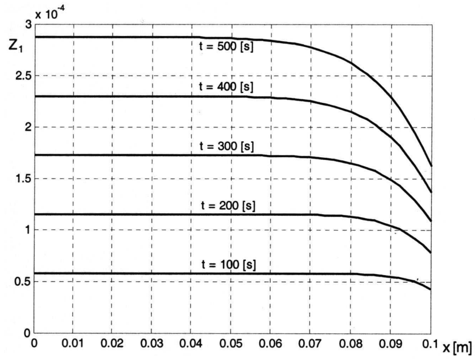


Fig. 2. Distribution of $U_1 = U_1(x, t)$

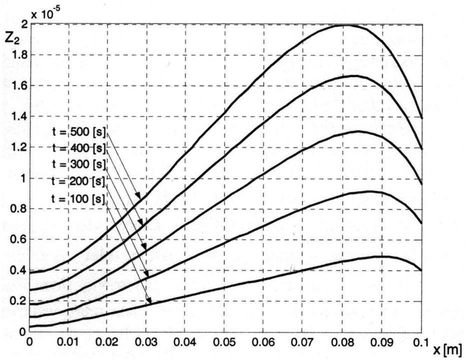


Fig. 3. Distribution of $U_2 = U_2(x, t)$

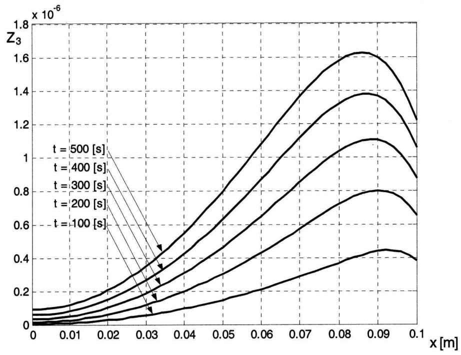
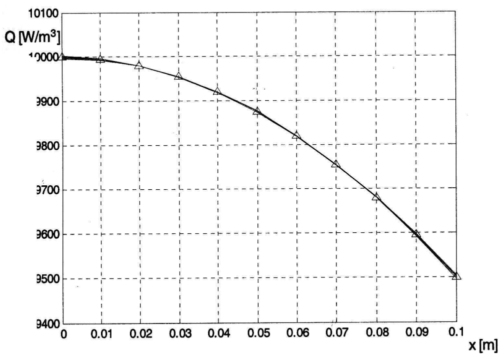


Fig. 4. Distribution of $U_3 = U_3(x, t)$



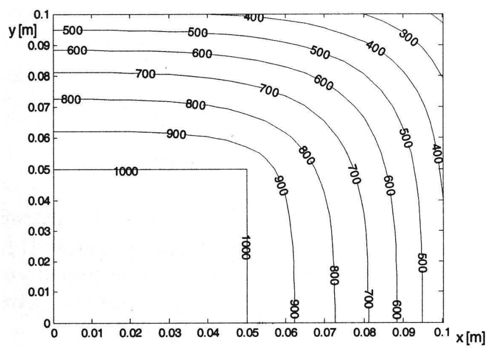


Fig. 5. Identification of $q(x)$

Fig. 6. Temperature distribution for t = 1000 s

References

[1] Kącki E., Równania różniczkowe cząstkowe w zagadnieniach fizyki i techniki, WNT, Warszawa 1992.

[2] Alifanov O.M., Inverse heat problems, Springer Verlag, Berlin 1994.

[3] Beck J.V., Blackwell B., Inverse problems, Handbook of Numerical Heat Transfer, Wiley Inters., New York 1988.

[4] Kurpisz K., Nowak A.J., Inverse thermal problems, Computational Mechanics Publications, Southampton, Boston 1995, 259-298.

[5] Metelski A., Identyfikacja wewnętrznych źródeł ciepła w procesie krzepnięcia odlewów, Doctoral Thesis, 2006.