IDENTIFICATION OF LATENT HEAT OF BIOLOGICAL TISSUE SUBJECTED TO THE FREEZING

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Abstract. In the paper the inverse problem consisting in the identification of volumetric latent heat of tissue subjected to the freezing is presented. Three different hypotheses concerning the substitute thermal capacity associated with the latent heat evolution are discussed. The 1D problem is considered and on the basis of the cooling curve on the skin surface the volumetric latent heat is estimated. In order to solve the inverse problem the least squares method containing the sensitivity coefficients is applied. In the final part of the paper the results of computations are shown.

1. Mathematical description of tissue freezing process

The equation describing the freezing process of biological tissue (1D problem) is of the form

$$0 < x < L: \quad c(T)\frac{\partial T(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[\lambda(T)\frac{\partial T(x,t)}{\partial x}\right] + Q\frac{\partial S(x,t)}{\partial t} \tag{1}$$

where $c(T)$ [W/(m^3 K)] is the volumetric specific heat of tissue, $\lambda(T)$ [W/(mK)] is the thermal conductivity, Q [J/(m^3 K)] is the volumetric latent heat, $S(x, t)$ is the volumetric frozen tissue fraction at the point considered, T, x, t denote the temperature, geometrical co-ordinate and time.

The source function in equation (1) can be eliminated (the one domain approach [1, 2]) and then one obtains (additionally assuming that $\lambda(T)$ = const)

$$0 < x < L: \quad C(T)\frac{\partial T(x,t)}{\partial t} = \lambda\frac{\partial^2 T(x,t)}{\partial x^2} \tag{2}$$

where

$$C(T) = \left[c(T) - Q\frac{\mathrm{d}S(T)}{\mathrm{d}T}\right] \tag{3}$$

is the substitute thermal capacity.

This function can be defined as follows

$$C(T)=\begin{cases} c_n, & T>T_b \\ \dfrac{c_n+c_f}{2}-Q\dfrac{\mathrm{d}S(T)}{\mathrm{d}T}, & T_e\le T\le T_b \\ c_f, & T<T_e \end{cases} \tag{4}$$

where the temperatures T_b, T_e correspond to the beginning and the end of the freezing process, respectively, c_n, c_f are the constant volumetric specific heats of natural and frozen tissue.

It should be pointed out that the function $C(T)$ fulfills the condition

$$\int_{T_e}^{T_b} C(T)\,\mathrm{d}T=\frac{c_f+c_n}{2}(T_b-T_e)+Q \tag{5}$$

Three hypotheses concerning the substitute thermal capacity $C(T)$ have been taken into account - Figure 1. If one assumes that for $T\in[T_e, T_b]$ $C(T)=c$, where c is a constant value then

$$T\in[T_e,T_b]:\quad C(T)=\frac{c_f+c_n}{2}+\frac{Q}{T_b-T_e} \tag{6}$$

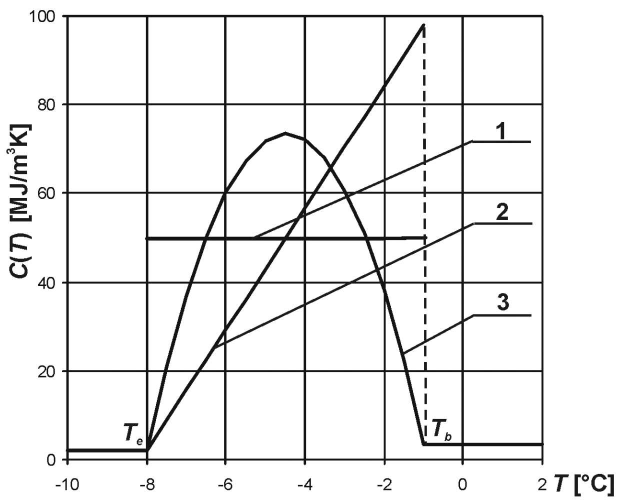

Fig. 1. Substitute thermal capacity (hypotheses 1, 2 and 3)

For linear function $C(T)=c_1+c_2T$, additionally assuming that $C(T_e)=c_f$ one obtains

$$T \in [T_e, T_b]: \quad C(T) = c_f + \left[\frac{c_n - c_f}{T_b - T_e} + \frac{2Q}{(T_b - T_e)^2} \right] \left[T(x,t) - T_e \right] \tag{7}$$

For function $C(T) = c_1 + c_2 T + c_3 T^2$ under the assumptions that $C(T_e) = c_f$ and $C(T_b) = c_n$ one has

$$\begin{gathered} T \in [T_e, T_b]: \quad C(T) = \frac{c_f T_b - c_n T_e}{T_b - T_e} + \frac{c_n - c_f}{T_b - T_e} T(x,t) + \\ \frac{6Q}{T_b^3 + 3T_b T_e^2 - 3T_b^2 T_e - T_e^3} \left[-T_b T_e + (T_b + T_e) T(x,t) - T(x,t)^2 \right] \end{gathered} \tag{8}$$

The equation (2) is supplemented by the initial condition

$$t = 0: \quad T(x,0) = T_0 \tag{9}$$

and boundary conditions:

$$\begin{aligned} x = 0: \quad & q(x,t) = \lambda \frac{\partial T(x,t)}{\partial x} = q_b \\ x = L: \quad & q(x,t) = -\lambda \frac{\partial T(x,t)}{\partial x} = 0 \end{aligned} \tag{10}$$

2. Inverse problem

We assume that the volumetric latent heat Q is unknown. If the inverse problem is formulated then it is necessary to have an additional information concerning the course of the process considered. So, let us assume that the course of surface temperature $T(0, t)$ is known

$$T_d^f = T\left(0, t^f\right), \quad f = 1, 2, \ldots, F \tag{11}$$

In order to solve the inverse problem, the least squares criterion is applied [3]

$$S(Q) = \frac{1}{F} \sum_{f=1}^{F} \left(T^f - T_d^f \right)^2 \tag{12}$$

where $T^f = T(0, t^f)$ is the calculated temperature at the point $x = 0$ for time t^f for arbitrary assumed value of Q.

The criterion (12) is differentiated with respect to the unknown volumetric latent heat Q and next the necessary condition of optimum is applied

$$\frac{\mathrm{d}S}{\mathrm{d}Q} = \frac{2}{F} \sum_{f=1}^{F} \left(T^f - T_d^f\right) \frac{\partial T^f}{\partial Q}\bigg|_{Q=Q^k} = 0 \tag{13}$$

where k is the number of iteration, Q^k for $k = 0$ is the arbitrary assumed value of Q, while Q^k for $k > 0$ results from the previous iteration.

Function T^f is expanded in a Taylor series about known value of Q^k, this means

$$T^f = \left(T^f\right)^k + \frac{\partial T^f}{\partial Q}\bigg|_{Q=Q^k} \left(Q^{k+1} - Q^k\right) \tag{14}$$

Putting (14) into (13) one has

$$\sum_{f=1}^{F} \left[\left(Z^f\right)^k\right]^2 \left(Q^{k+1} - Q^k\right) = \sum_{f=1}^{F} \left(Z^f\right)^k \left[T_d^f - \left(T^f\right)^k\right] \tag{15}$$

this means

$$Q^{k+1} = Q^k + \frac{\sum_{f=1}^{F} \left(Z^f\right)^k \left[T_d^f - \left(T^f\right)^k\right]}{\sum_{f=1}^{F} \left[\left(Z^f\right)^k\right]^2}, \qquad k = 0, 1, \ldots, K \tag{16}$$

where

$$\left(Z^f\right)^k = \frac{\partial T^f}{\partial Q}\bigg|_{Q=Q^k} \tag{17}$$

are the sensitivity coefficients. Equation (16) allows to find the value of Q^{k+1}. The iteration process is stopped when the assumed accuracy is achieved or after the achieving the assumed value of iterations.

In order to determine the sensitivity coefficients (17), the governing equations (2), (9), (10) should be differentiated with respect to the unknown parameter Q [4]. So, the differentiation of equation (2) leads to the formula

$$C(T)\frac{\partial Z(x,t)}{\partial t} = \lambda \frac{\partial^2 Z(x,t)}{\partial x^2} - \frac{\partial C(T)}{\partial Q}\frac{\partial T(x,t)}{\partial t} \tag{18}$$

while differentiation of conditions (9), (10) gives:

$$\begin{aligned} x = 0: &\quad \lambda \frac{\partial Z(x,t)}{\partial x} = 0 \\ x = L: &\quad -\lambda \frac{\partial Z(x,t)}{\partial x} = 0 \\ t = 0: &\quad Z(x,0) = 0 \end{aligned} \tag{19}$$

Taking into account the formula (6) one has

$$\frac{\partial C(T)}{\partial Q}=\begin{cases}0, & T>T_b\\ \dfrac{1}{T_b-T_e}, & T_e\le T\le T_b\\ 0, & T<T_e\end{cases} \tag{20}$$

while using the formula (7) one obtains

$$\frac{\partial C(T)}{\partial Q}=\begin{cases}0, & T>T_b\\ \left[\dfrac{c_n-c_f}{T_b-T_e}+\dfrac{2Q}{(T_b-T_e)^2}\right]Z(x,t) & \\ +\dfrac{2}{(T_b-T_e)^2}\left[T(x,t)-T_e\right], & T_e\le T\le T_b\\ 0, & T<T_e\end{cases} \tag{21}$$

and for the formula (8)

$$\frac{\partial C(T)}{\partial Q}=\begin{cases}0, & T>T_b\\ \dfrac{c_n-c_f}{T_b-T_e}Z(x,t)+\dfrac{6}{T_b^3+3T_bT_e^2-3T_b^2T_e-T_e^3}\cdot & \\ \left\{\left[-T_bT_e+(T_b+T_e)T(x,t)-T(x,t)^2\right]+\right. & T_e\le T\le T_b\\ \left.Q\left[(T_b+T_e)Z(x,t)-2T(x,t)Z(x,t)\right]\right\}, & \\ 0, & T<T_e\end{cases} \tag{22}$$

3. Results of computations

The tissue of thickness L = 0.02 m has been considered. The following data have been assumed: λ = 1.26 W/(mK), c_n = 3.6 MW/(m^{3}K), c_f = 1.93 MW/(m^{3}K), T_b = –1°C, T_e = –8°C, initial temperature T_0 = 37°C, for time $t \le 180$ s boundary heat flux equals q_b = 10000 W/m^2, for time $t > 180$ s, q_b = –5000 W/m^2.

The basic problem and additional one connected with the sensitivity function have been solved using the explicit scheme of finite differences method [1] with time step Δt = 0.01 s and mesh step h = 0.0002 m.

In Figure 2 the cooling (heating) curves at the point x = 0 for different hypotheses of substitute thermal capacity are shown. They result from the direct

problem solution under the assumption that $Q = 330$ MJ/(m^{3}K). Figure 3 illustrates the
courses of sensitivity function $Z(x, t)$ for $x = 0$, different hypotheses of $C(T)$ and real value of Q.

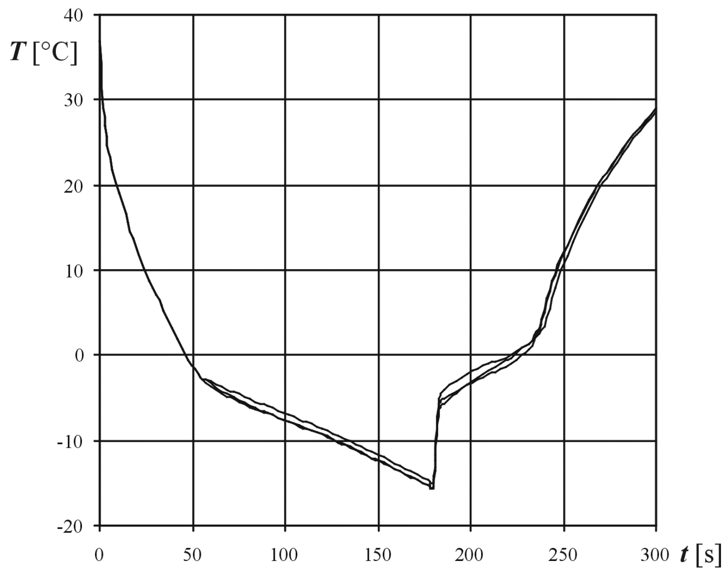

Fig. 2. Cooling curves for $x = 0$ and different hypotheses

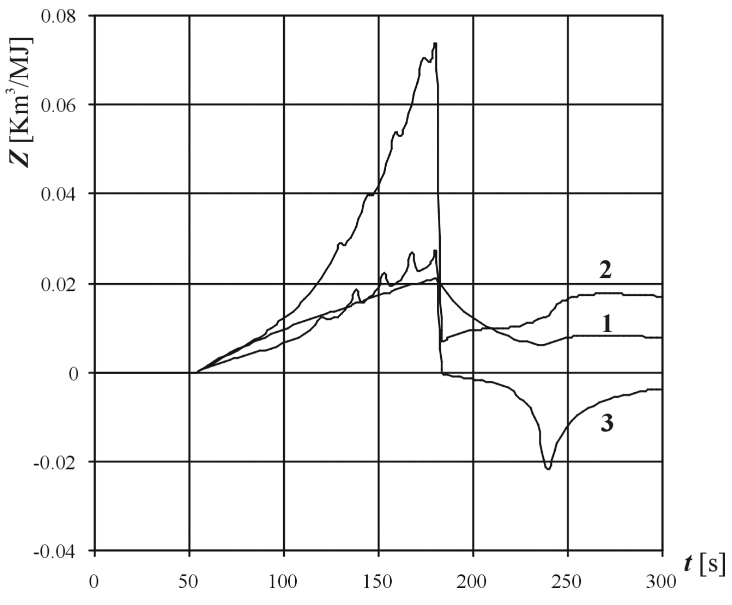

Fig. 3. Courses of function Z for x = 0 and different hypotheses

Figure 4 illustrates the results of substitute thermal capacity identification for initial value $Q^0 = 0$ and cooling (heating) curves presented in Figure 2. It is visible that the iteration process is quickly convergent. Figure 5 shows the courses of function S (c.f. equation (12)) for successive iterations and hypotheses 1, 2, 3.

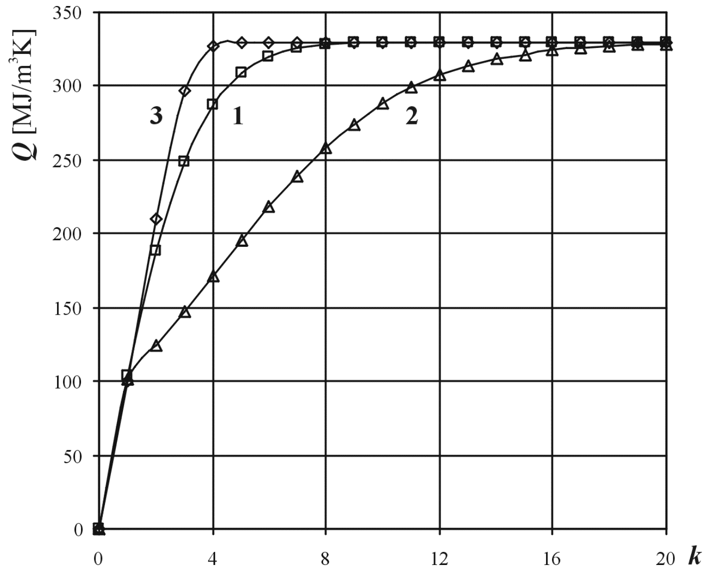

Fig. 4. Identification of Q for different hypotheses

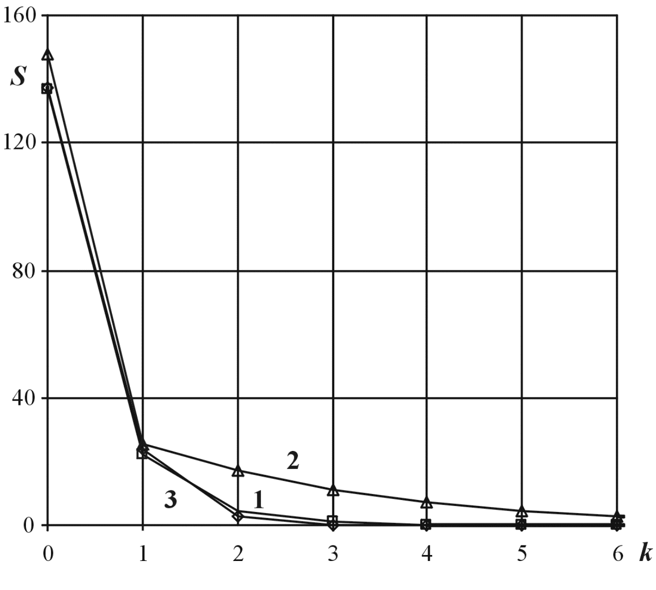

Fig. 5. Values of function S for different hypotheses

Summing up, for assumed initial value of Q and hypothesis 1 the iteration process gives the real value of Q after 9 iterations, for hypothesis 2 after 20 iterations and for hypothesis 3 after 5 iterations (Fig. 4).

The paper has been sponsored by KBN (Grant No 3 T11F 018 26).

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