COMPUTER IMPLEMENTATION OF THE DUAL RECIPROCITY BEM FOR 2D POISSON'S EQUATION

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Abstract. In this paper the variant of the boundary element method called dual reciprocity BEM is presented. On the stage of numerical computations the DRBEM application for the Poisson equation allows to avoid the discretization of the interior of the domain considered. In the final part of the paper the results of computations are shown.

1. Governing equations

We consider the Poisson equation

$$(x,\, y) \in \Omega: \quad \lambda \nabla^2 T(x,\, y) + Q(x,\, y) = 0 \tag{1}$$

where λ [W/mK] is the thermal conductivity, T is the temperature, x, y are the geometrical co-ordinates, $Q(x, y)$ [W/m^3] is the source function. The equation (1) is supplemented by boundary conditions:

$$\begin{aligned} (x,\, y) \in \Gamma_1: &\quad T(x,\, y) = T_b \\ (x,\, y) \in \Gamma_2: &\quad q(x,\, y) = -\lambda \mathbf{n} \cdot \nabla T(x,\, y) = q_b \end{aligned} \tag{2}$$

where T_b is known boundary temperature, $\mathbf{n}$ is the normal outward vector at the boundary point (x, y), q_b is given boundary heat flux.

2. Boundary element method for the Poisson equation

The integral equation for problem (1), (2) is following [1, 2]

$$\begin{gathered} B(\xi,\, \eta) T(\xi,\, \eta) + \int_\Gamma T^*(\xi,\, \eta,\, x,\, y) q(x,\, y) \mathrm{d}\Gamma = \\ \int_\Gamma q^*(\xi,\, \eta,\, x,\, y) T(x,\, y) \mathrm{d}\Gamma + \iint_\Omega T^*(\xi,\, \eta,\, x,\, y) Q(x,\, y) \mathrm{d}\Omega \end{gathered} \tag{3}$$

where (ξ, η) is the observation point (source point), if $(\xi,\eta)\in\Gamma$ then $B(\xi,\eta)$ is the coefficient connected with the local shape of boundary, if $(\xi,\eta)\in\Omega$ then $B(\xi,\eta) = 1$, $T^*(\xi, \eta, x, y)$ is the fundamental solution, while

$$q^*(\xi, \eta, x, y) = -\lambda \mathbf{n}\cdot\nabla T^*(\xi, \eta, x, y) \tag{4}$$

and

$$q(x, y) = -\lambda \mathbf{n}\cdot\nabla T(x, y) \tag{5}$$

Fundamental solution has the following form

$$T^*(\xi, \eta, x, y) = \frac{1}{2\pi\lambda}\ln\frac{1}{r} \tag{6}$$

where r is the distance between the points (ξ, η) and (x, y)

$$r = \sqrt{(x-\xi)^2 + (y-\eta)^2} \tag{7}$$

It should be pointed out that the function $T^*(\xi, \eta, x, y)$ fulfills the equation

$$\lambda\nabla^2 T^*(\xi, \eta, x, y) = -\delta(\xi, \eta, x, y) \tag{8}$$

where $\delta(\xi, \eta, x, y)$ is the Dirac function.

Heat flux resulting from the fundamental solution can be calculated analytically and then

$$q^*(\xi, \eta, x, y) = \frac{d}{2\pi r^2} \tag{9}$$

where

$$d = (x-\xi)\cos\alpha + (y-\eta)\cos\beta \tag{10}$$

while $\cos\alpha$, $\cos\beta$ are the directional cosines of the boundary normal vector $\mathbf{n}$.

3. Dual reciprocity BEM for the Poisson equation

The solution of Poisson's equation (1) can be written as a sum [3]

$$T(x, y) = \hat{T}(x, y) + U(x, y) \tag{11}$$

where the first function is the solution of Laplace's equation

$$(x,\, y) \in \Omega: \quad \lambda \nabla^2 \hat{T}(x,\, y) = 0 \tag{12}$$

while $U(x, y)$ is a particular solution

$$(x,\, y) \in \Omega: \quad \lambda \nabla^2 U(x,\, y) = -Q(x,\, y) \tag{13}$$

It is generally difficult to find a solution $U(x, y)$, so in the dual reciprocity method the following approximation for $Q(x, y)$ is proposed [3]

$$Q(x,\, y) \approx \sum_{k=1}^{N+L} a_k f_k(x,\, y) \tag{14}$$

where a_k are unknown coefficients and $f_k(x,\, y)$ are approximating functions fulfilling the equations

$$-\lambda \nabla^2 U_k(x,\, y) = f_k(x,\, y) \tag{15}$$

In equation (14) $N+L$ corresponds to the total number of nodes, where N is the number of boundary nodes and L is the number of internal nodes.
Putting (15) into (14) one obtains

$$Q(x,\, y) = -\lambda \sum_{k=1}^{N+L} a_k \nabla^2 U_k(x,\, y) \tag{16}$$

We consider the last integral in equation (3)

$$\begin{gathered} D = \iint_{\Omega} T^*(\xi,\, \eta,\, x,\, y) Q(x,\, y) \mathrm{d}\Omega = \\ -\sum_{k=1}^{N+L} a_k \iint_{\Omega} \left[\lambda \nabla^2 U_k(x,\, y) \right] T^*(\xi,\, \eta,\, x,\, y) \mathrm{d}\Omega \end{gathered} \tag{17}$$

Using the second Green formula one has

$$\begin{gathered} D = -\sum_{k=1}^{N+L} a_k \iint_{\Omega} \left[\lambda \nabla^2 T^*(\xi,\, \eta,\, x,\, y) \right] U_k(x,\, y) \mathrm{d}\Omega - \\ \sum_{k=1}^{N+L} a_k \int_{\Gamma} \left[\lambda T^*(\xi,\, \eta,\, x,\, y) \mathbf{n} \cdot \nabla U_k(x,\, y) - \right. \\ \left. \lambda U_k(x,\, y) \mathbf{n} \cdot \nabla T^*(\xi,\, \eta,\, x,\, y) \right] \mathrm{d}\Gamma \end{gathered} \tag{18}$$

Because (c.f. formula (8))

$$\begin{gathered}\iint_{\Omega}\left[\lambda\nabla^2 T^*(\xi,\eta,x,y)\right]U_k(x,y)\,\mathrm{d}\Omega = \\ -\iint_{\Omega}\delta(\xi,\eta,x,y)U_k(x,y)\,\mathrm{d}\Omega = -B(\xi,\eta)U_k(\xi,\eta)\end{gathered} \tag{19}$$

so

$$\begin{gathered}D = \sum_{k=1}^{N+L} a_k \Big[B(\xi,\eta)U_k(\xi,\eta) + \\ \int_{\Gamma} T^*(\xi,\eta,x,y)W_k(x,y)\,\mathrm{d}\Gamma - \int_{\Gamma} U_k(x,y)q^*(\xi,\eta,x,y)\,\mathrm{d}\Gamma\Big]\end{gathered} \tag{20}$$

where

$$W_k(x,y) = -\lambda\mathbf{n}\cdot\nabla U_k(x,y) \tag{21}$$

Taking into account the formula (20) the equation (3) can be written in the form

$$\begin{gathered}B(\xi,\eta)T(\xi,\eta) + \int_{\Gamma} T^*(\xi,\eta,x,y)q(x,y)\,\mathrm{d}\Gamma = \\ \int_{\Gamma} q^*(\xi,\eta,x,y)T(x,y)\,\mathrm{d}\Gamma + \sum_{k=1}^{N+L} a_k\Big[B(\xi,\eta)U_k(\xi,\eta) + \\ \int_{\Gamma} T^*(\xi,\eta,x,y)W_k(x,y)\,\mathrm{d}\Gamma - \int_{\Gamma} q^*(\xi,\eta,x,y)U_k(x,y)\,\mathrm{d}\Gamma\Big]\end{gathered} \tag{22}$$

4. Numerical realization of DRBEM

In order to solve the equation (22), the boundary Γ is divided into N boundary elements and in the interior of the domain L internal nodes are distinguished. In the case of constant boundary elements we assume that

$$(x,y)\in\Gamma_j: \quad \begin{cases} T(x,y) = T(x_j,y_j) = T_j \\ q(x,y) = q(x_j,y_j) = q_j \end{cases} \tag{23}$$

and

$$(x,y)\in\Gamma_j: \quad \begin{cases} U_k(x,y) = U_k(x_j,y_j) = U_{jk} \\ W_k(x,y) = W_k(x_j,y_j) = W_{jk} \end{cases} \tag{24}$$

So, the following approximation of equation (22) can be taken into account ($i = 1, 2, \ldots, N, N+1, \ldots, N+L$)

$$B_i T_i + \sum_{j=1}^{N} q_j \int_{\Gamma_j} T^*(\xi_i, \eta_i, x, y)\,\mathrm{d}\Gamma_j =$$
$$\sum_{j=1}^{N} T_j \int_{\Gamma_j} q^*(\xi_i, \eta_i, x, y)\,\mathrm{d}\Gamma_j + \sum_{k=1}^{N+L} a_k \Big[B_i U_{ik} + \tag{25}$$
$$\sum_{j=1}^{N} W_{jk} \int_{\Gamma_j} T^*(\xi_i, \eta_i, x, y)\,\mathrm{d}\Gamma_j - \sum_{j=1}^{N} U_{jk} \int_{\Gamma_j} q^*(\xi_i, \eta_i, x, y)\,\mathrm{d}\Gamma_j \Big]$$

or

$$B_i T_i + \sum_{j=1}^{N} G_{ij} q_j = \sum_{j=1}^{N} \hat{H}_{ij} T_j + \sum_{k=1}^{N+L} a_k \left(B_i U_{ik} + \sum_{j=1}^{N} G_{ij} W_{jk} - \sum_{j=1}^{N} \hat{H}_{ij} U_{jk} \right) \tag{26}$$

where

$$G_{ij} = \int_{\Gamma_j} T^*(\xi_i, \eta_i, x, y)\,\mathrm{d}\Gamma_j \tag{27}$$

and

$$\hat{H}_{ij} = \int_{\Gamma_j} q^*(\xi_i, \eta_i, x, y)\,\mathrm{d}\Gamma_j \tag{28}$$

while $B_i = B(\xi_i, \eta_i)$. We define [3, 4]

$$U_{jk} = \frac{r_{jk}^2}{4} + \frac{r_{jk}^3}{9} \tag{29}$$

where (Figure 1)

$$r_{jk}^2 = (x_k - x_j)^2 + (y_k - y_j)^2 \tag{30}$$

Using the formula (21) one obtains

$$W_{jk} = -\lambda \begin{bmatrix} \cos\alpha_j & \cos\beta_j \end{bmatrix} \begin{bmatrix} \dfrac{\partial U_{jk}}{\partial x_j} \\ \dfrac{\partial U_{jk}}{\partial y_j} \end{bmatrix} = \lambda d_{jk} \left(\frac{1}{2} + \frac{1}{3} r_{jk} \right) \tag{31}$$

where

$$d_{jk} = (x_k - x_j)\cos\alpha_j + (y_k - y_j)\cos\beta_j \tag{32}$$

Because

$$\nabla^2 U_{sk} = \frac{\partial^2 U_{sk}}{\partial x_k^2} + \frac{\partial^2 U_{sk}}{\partial y_k^2} = 1 + r_{sk} \tag{33}$$

so on the basis of equation (15) one has

$$f_{sk} = f_k(x_s, y_s) = -\lambda(1 + r_{sk}) \tag{34}$$

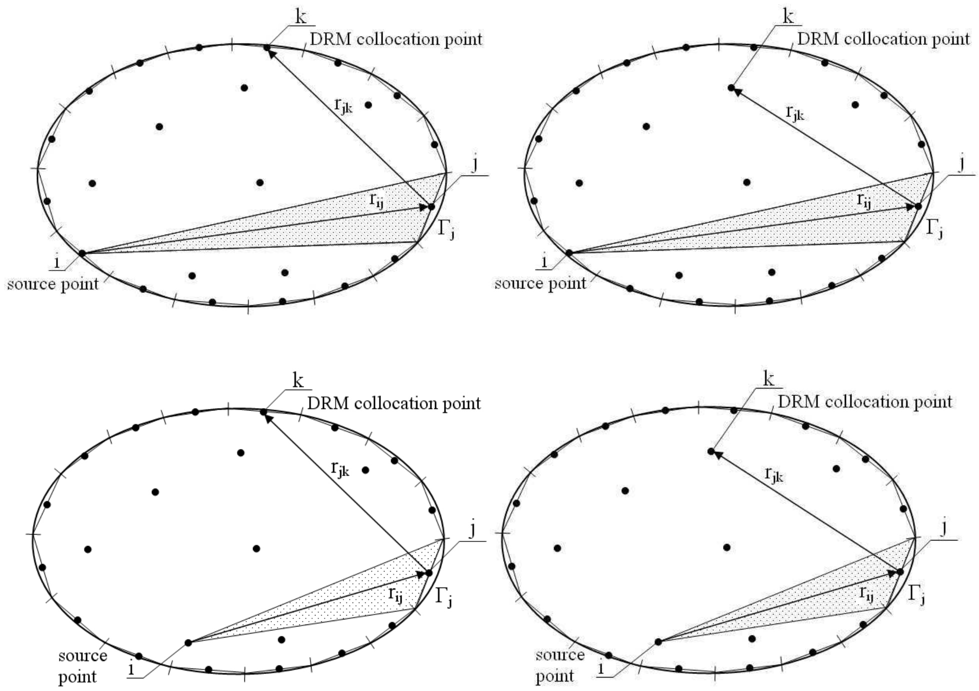

Fig. 1. Vectors r_{ij} and r_{jk} for boundary and internal source points

The equation (16) can be expressed as follows

$$Q_s = Q(x_s, y_s) = -\lambda \sum_{k=1}^{N+L} a_k (1 + r_{sk}), \qquad s = 1, 2, \ldots, N + L \tag{35}$$

The system of equations (35) can be written in the matrix form

$$\begin{bmatrix} Q_1 \\ Q_2 \\ \cdots \\ Q_{N+L} \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1,N+L} \\ f_{21} & f_{22} & \cdots & f_{2,N+L} \\ \cdots & \cdots & \cdots & \cdots \\ f_{N+L,1} & f_{N+L,2} & \cdots & f_{N+L,N} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \cdots \\ a_{N+L} \end{bmatrix} \tag{36}$$

or

$$\mathbf{Q} = \mathbf{f}\,\mathbf{a} \tag{37}$$

The following matrices of dimensions $(N+L) \times (N+L)$ can be defined

$$\mathbf{G} = \begin{bmatrix} G_{11} & G_{12} & \cdots & G_{1,N} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ G_{N,1} & G_{N,2} & \cdots & G_{N,N} & 0 & \cdots & 0 \\ G_{N+1,1} & G_{N+1,2} & \cdots & G_{N+1,N} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ G_{N+L,1} & G_{N+L,2} & \cdots & G_{N+L,N} & 0 & \cdots & 0 \end{bmatrix} \tag{38}$$

$$\mathbf{H} = \begin{bmatrix} H_{11} & H_{12} & \cdots & H_{1,N} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ H_{N,1} & H_{N,2} & \cdots & H_{N,N} & 0 & \cdots & 0 \\ H_{N+1,1} & H_{N+1,2} & \cdots & H_{N+1,N} & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ H_{N+L,1} & H_{N+L,2} & \cdots & H_{N+L,N} & 0 & \cdots & -1 \end{bmatrix} \tag{39}$$

where

$$H_{ij} = \begin{cases} \hat{H}_{ij}, & i \neq j \\ \hat{H}_{ij} - 1/2, & i = j \end{cases} \tag{40}$$

and

$$\mathbf{U} = \begin{bmatrix} U_{11} & U_{12} & \cdots & U_{1,N} & U_{1,N+1} & \cdots & U_{1,N+L} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ U_{N,1} & U_{N,2} & \cdots & U_{N,N} & U_{N,N+1} & \cdots & U_{N,N+L} \\ U_{N+1,1} & U_{N+1,2} & \cdots & U_{N+1,N} & U_{N+1,N+1} & \cdots & U_{N+1,N+L} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ U_{N+L,1} & U_{N+L,2} & \cdots & U_{N+L,N} & U_{N+L,N+1} & \cdots & U_{N+L,N+L} \end{bmatrix} \tag{41}$$

$$\mathbf{W} = \begin{bmatrix} W_{11} & W_{12} & \dots & W_{1,N} & W_{1,N+1} & \dots & W_{1,N+L} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ W_{N,1} & W_{N,2} & \dots & W_{N,N} & W_{N,N+1} & \dots & W_{N,N+L} \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix} \tag{42}$$

So, the system of equations (26) can be written in the matrix form

$$\mathbf{Gq} = \mathbf{HT} + (\mathbf{GW} - \mathbf{HU})\mathbf{a} \tag{43}$$

or (c.f. formula (37))

$$\mathbf{Gq} = \mathbf{HT} + (\mathbf{GW} - \mathbf{HU})\mathbf{f}^{-1}\mathbf{Q} \tag{44}$$

where

$$\mathbf{T} = \begin{bmatrix} T_1 \\ \dots \\ T_N \\ T_{N+1} \\ \dots \\ T_{N+L} \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} q_1 \\ \dots \\ q_N \\ 0 \\ \dots \\ 0 \end{bmatrix} \tag{45}$$

5. Results of computations

The square of dimensions 1×1 has been considered. Thermal conductivity equals $\lambda = 1$. On the left and bottom parts of the boundary the Neumann condition $q_b = 0$ has been assumed, on the remaining parts of the boundary the Dirichlet condition $T_b = 0$ has been accepted.

The problem has been solved using classical algorithm of the BEM (c.f. chapter 2) and then the boundary has been divided into 20 constant boundary elements and the interior has been divided into 25 constant internal cells. The results obtained for $Q = 100$ (variant 1) and $Q = 1000(x^3 + y^3)$ (variant 2) are shown in Figure 2.

The same problems have been solved using dual reciprocity BEM for 20 constant boundary elements and different numbers of internal points. In Figures 3, 4, 5 and 6 the results of computations are shown.

It is visible that in the case of constant function Q the number of internal points is not especially essential, but in the case $Q = Q(x, y)$ even for big number of internal points the results are still not acceptable (Figures 2 and 6).

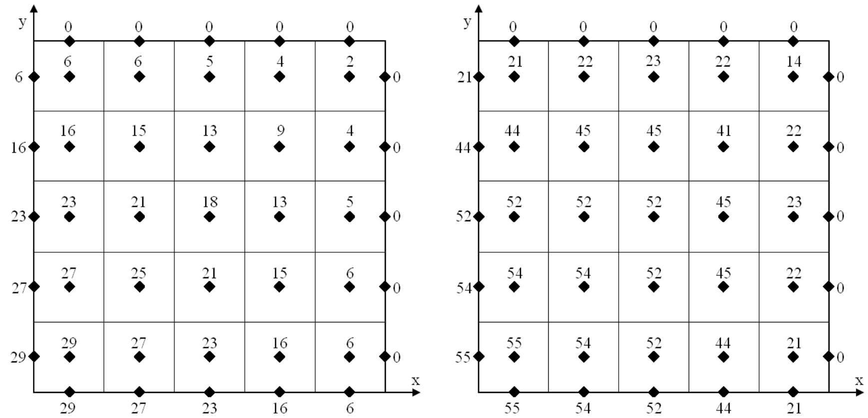

Fig. 2. BEM solution for Q = 100 (variant 1) and $Q = 1000(x^3 + y^3)$ (variant 2)

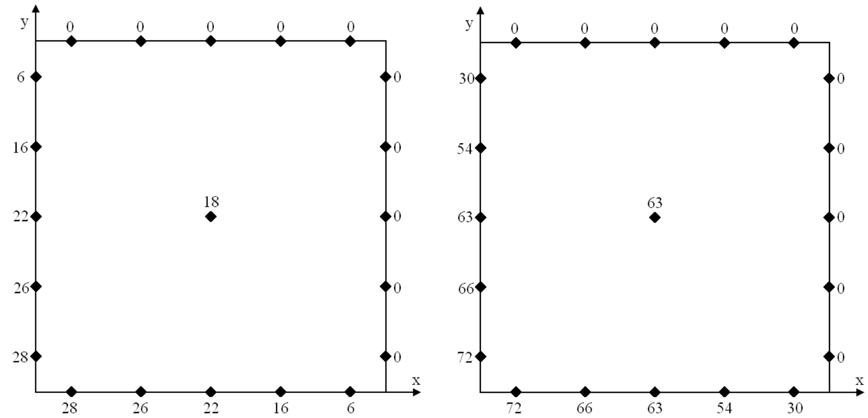

Fig. 3. DRBEM solution for variants 1 and 2 - one internal point

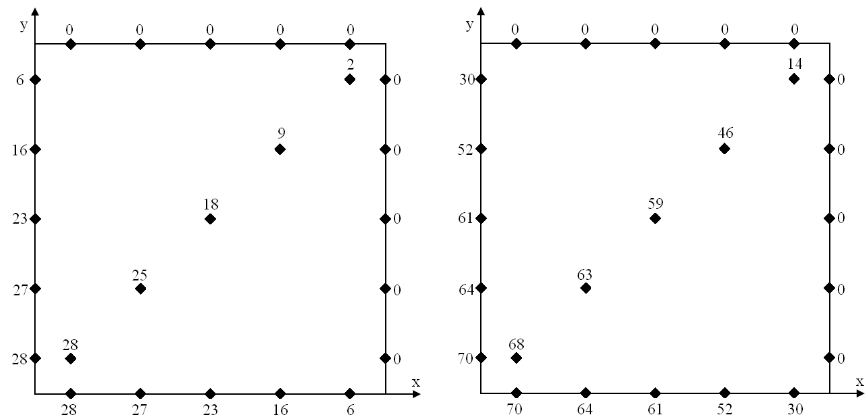

Fig. 4. DRBEM solution for variants 1 and 2 - five internal points

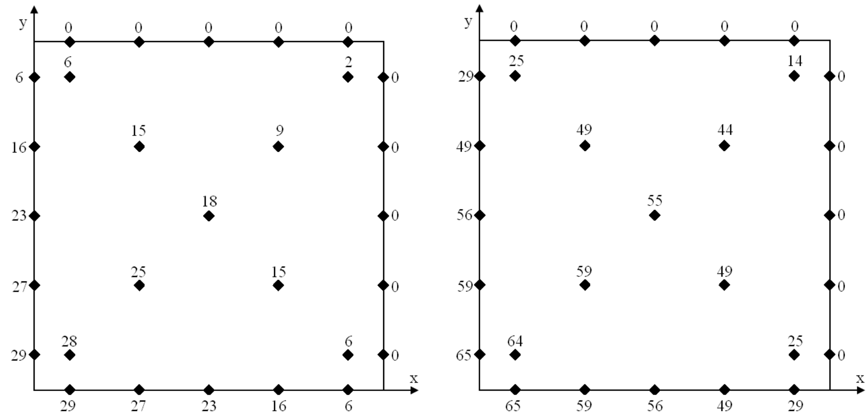

Fig. 5. DRBEM solution for variants 1 and 2 - nine internal points

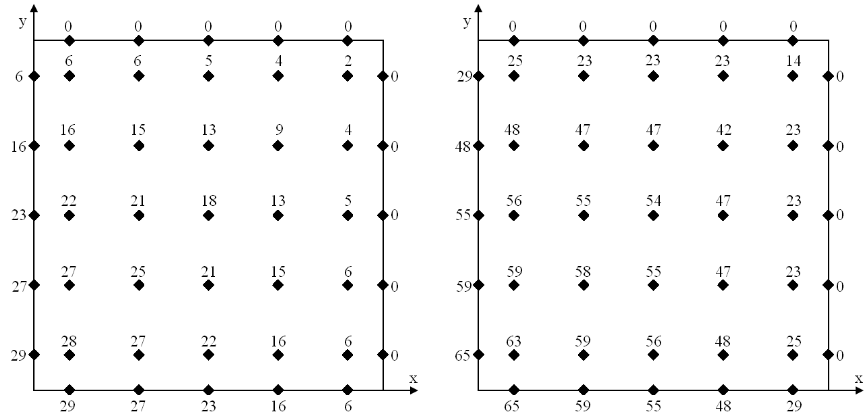

Fig. 6. DRBEM solution for variants 1 and 2 - twenty five internal points

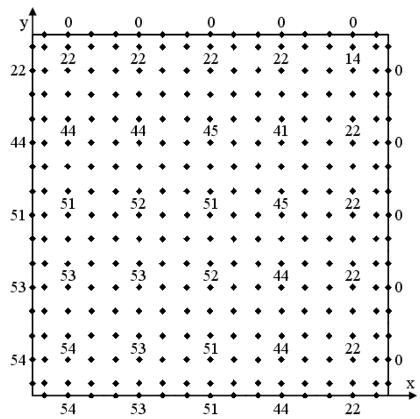

Fig. 7. DRBEM solution - variant 2, N = 60, L = 225

It should be pointed out that for source function $Q(x, y) = 1000(x^3 + y^3)$ and $N = 60$ constant boundary elements and $L = 225$ internal points the results obtained using DRBEM are almost the same as in the case of classical BEM application - Figure 7.

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