ON THE PROBABILISTIC ANALOGY BETWEEN HOMOGENIZED AND TOLERANCE MODELS OF TWO-CONSTITUENT MICROPERIODIC ISOTROPIC LAMINATED COMPOSITES

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Abstract. In the note an interrelation between tolerance and homogenized effective moduli for two-constituent elastic microperiodic composite are examined. It is shown that, in the case in which constituents are isotropic, effective modulae are identical. The proof is based on the probabilistic analogy formulated for both kinds of modulae.

Introduction

In many problems of mechanics in which we deal with macroscopic description of periodic microstructured media a certain averaged mathematical objects should be applied. This averaged objects are defined in different ways but these definitions always depends on the properties of the model of the microstuctured media which is taken into account. There exist situations in which different methods of modelling lead to the models with the same mathematical form (in a certain special cases) and coefficients defined on the quite different way. With such situations we deal with for models obtained by an application of the both homogenization and tolerance averaging techniques. Usually such kind of models lead to different or identical descriptions of the same physical situation and lead to very closed or quite different solutions to the same problem. The aim of this note is to investigate this problem for static problems of linear elasticity.

1. Formulation of the problem

Let us consider linear elastic two-constituent laminated microperiodic composite. By l we denote the thickness of the repeated layer and by η' and η'' factors of two laminae. Hence $\eta' l$ and $\eta'' l$ are lengths of laminae. Moreover by ρ' and ρ'' and C'_{ijkl} and C''_{ijkl} are mass densities and elastic modulus, respectively. Under the assumption that Ox_3-axis of the Carthesian orthogonal coordinate system $Ox_1x_2x_3$ determines the periodicity direction, we are to shortly describe two models of linear elastic two-constituent laminated microperiodic composites: tolerance averaged model and homogenized model.

Tolerance averaged model

The tolerance averaged model of linear elastic two-constituent laminated microperiodic composite can be represented by the system of equations, cf. [1]

$$\begin{aligned} &\langle \rho \rangle \ddot{U}_i - \langle C_{ijkl} \rangle U_{i,jk} = \langle C_{i33l} g^B{}_{,3} \rangle \zeta^B{}_{k,3} \\ &l^2 [\langle \rho g^A g^B \rangle \ddot{\zeta}^B_i - \langle C_{i33l} g^A g^B \rangle \zeta^B{}_{l,33}] + \langle C_{i33l} g^A{}_{,3}\, g^B{}_{,3} \rangle \zeta^B{}_l + \langle C_{i33l} g^B{}_{,3} \rangle U_{l,3} = 0 \quad (1) \\ &\qquad\qquad A, B = 1, \ldots, N \end{aligned}$$

where g^A, $A = 1,\ldots,N$, are shape functions. The basic unknowns of the tolerance model are:

1° The averaged displacement field U_i defined by

$$U_i = \langle u_i \rangle \tag{2}$$

where for integrable function f the averaged operator is taken as

$$\langle f \rangle (x_1, x_2, x_3) = \frac{1}{l} \int_{x_3 - l/2}^{x_3 + l/2} f(x_1, x_2, z) dz \tag{3}$$

2° Internal variables ζ_i^A, $A = 1,\ldots,N$, $i = 1,2,3$, which are introduced by an additional assumption that the residual displacements r_i can be approximated by the finite sum

$$r_i \equiv u_i - U_i \cong g^A \zeta_i^A \tag{4}$$

The above basic unknowns should be restricted by the conditions of the physical correctness of the tolerance model, which will be written in the form

$$U_i, \zeta_i^A \in SV_l(T) \tag{5}$$

where $A = 1,\ldots,N$, $i = 1,2,3$. The form and the number N of shape functions are postulated a priori in every special problem. In the asymptotic case, $l \to 0$, under additional assumption that basic model unknowns are slowly varying in all directions model equations system (1) reduces to the form

$$\begin{aligned} &\langle \rho \rangle \ddot{U}_i - \langle C_{ijkl} \rangle U_{l,jk} = \langle C_{i33l} g^B{}_{,3} \rangle \zeta^B{}_{l,3} \\ &\langle C_{i33l} g^A{}_{,3}\, g^B{}_{,3} \rangle \zeta^B{}_l + \langle C_{i33l} g^B{}_{,3} \rangle U_{l,3} = 0 \\ &\qquad\qquad A, B = 1, \ldots, N \end{aligned} \tag{6}$$

The characteristic feature of the above system is a possibility of elimination internal variables ζ_i^A, $A = 1,\ldots,N$, $i = 1,2,3$, from the second of model equations (6) and in this case (6) reduces to the single equation

$$\langle\rho\rangle\ddot{U}_i - C^{eff}_{ijkl}U_{l},_{kj} = 0 \tag{7}$$

where

$$C^{eff}_{ijkl} = \langle C_{jkl}\rangle - \langle C_{ij3p}g^A,_3\rangle H^{AB}_{pq}\langle C_{q3kl}g^B,_3\rangle \qquad A,B = 1,\ldots,N \tag{8}$$

is the tolerance effective modulus tensor.

In the case of two-constituent periodic laminated conductor, which is illustrated in Figure 1, the shape function system consists of exclusively one saw-like shape function illustrated in Figure 1.

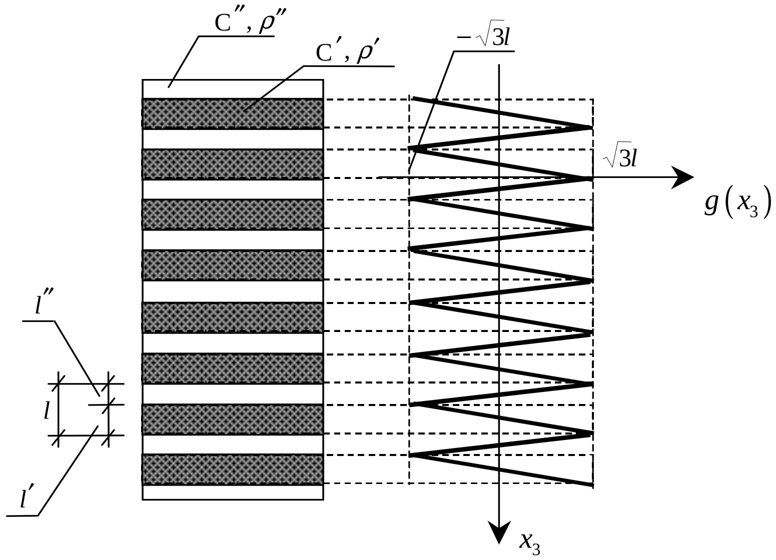

Fig. 1. A fragment of a laminated two-constituent laminated rigid composite solid together with the diagram of applied shape function

In this case the system (1) takes the form:

$$\begin{aligned}
&\langle\rho\rangle\ddot{U}_i - \langle C_{ijkl}\rangle U_{l},_{jk} = [C_{i33l}]\zeta_{k},_3 \\
&l^2(\langle\rho\rangle\ddot{\zeta}_i - \langle C_{i\beta\gamma l}\rangle\zeta_{l},_{\beta\gamma}) + \{C_{i33l}\}\zeta_l + [C_{i33l}]U_{l},_3 = 0
\end{aligned} \tag{9}$$

where:

$$\begin{aligned} &\langle C\rangle = \eta' C' + \eta'' C'' \\ &\{C\} = \langle C g,_3 g,_3 \rangle = 12\left(\frac{C'_{i33l}}{\eta_1} + \frac{C''_{i33l}}{\eta_2}\right) \\ &[C] = \langle C g,_3 \rangle = C'_{ij3l} - C''_{ij3l} \end{aligned} \tag{10}$$

In the case of two-constituent laminated composite (8) takes the form

$$C^{eff}_{ijkl} = \left\langle C_{ijkl} \right\rangle - [C]_{ijp} H_{pq} [C]_{klq} \tag{11}$$

Related to the aim of this note effective modulus (11) should be rewritten in more detailed form:

$$\begin{aligned} C^{o}_{1111} &= \langle C_{1111}\rangle - [C_{1111}]^2 / \{C_{1111}\} \\ C^{o}_{2222} &= \langle C_{2222}\rangle - [C_{2211}]^2 / \{C_{1111}\} \\ C^{o}_{3333} &= \langle C_{3333}\rangle - [C_{3311}]^2 / \{C_{1111}\} \\ C^{o}_{2211} &= \langle C_{2211}\rangle - [C_{2211}][C_{1111}] / \{C_{1111}\} \\ C^{o}_{3311} &= \langle C_{3311}\rangle - [C_{3311}][C_{1111}] / \{C_{1111}\} \\ C^{o}_{2233} &= \langle C_{2233}\rangle - [C_{2211}][C_{1133}] / \{C_{1111}\} \\ C^{o}_{1212} &= \langle C_{1212}\rangle - [C_{1212}]^2 / \{C_{1111}\} \\ C^{o}_{1313} &= \langle C_{1313}\rangle - [C_{3311}][C_{1111}] / \{C_{1111}\} \\ C^{o}_{2323} &= \langle C_{2323}\rangle \end{aligned} \tag{12}$$

where, similarly to (10):

$$\begin{aligned} &\langle C\rangle \equiv \eta' C' + \eta'' C'' \\ &\{C\} \equiv 12\left(\frac{C'}{\eta'} + \frac{C''}{\eta''}\right) \\ &[C] \equiv C' - C'' \end{aligned} \tag{13}$$

In the special case in which constituents of considered laminated composite are isotropic, i.e. elastic modulus are given by:

$$\begin{aligned} C'_{ijkl} &= \lambda' \delta_{ij}\delta_{kl} + \mu'(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \\ C''_{ijkl} &= \lambda'' \delta_{ij}\delta_{kl} + \mu''(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \end{aligned} \tag{14}$$

formulas (10) arrive at:

$$\begin{aligned} \{C\}_{ij} &= \{\lambda\}\delta_{i3}\delta_{3l} + \{\mu\}(\delta_{i3}\delta_{3l} + \delta_{il}) \\ [C]_{ijl} &= \{\lambda\}\delta_{ij}\delta_{3l} + \{\mu\}(\delta_{i3}\delta_{jl} + \delta_{il}\delta_{j3}) \end{aligned} \tag{15}$$

where:

$$\begin{aligned} \{\lambda\} &= \langle \lambda g,_3 g,_3 \rangle = 12\left(\frac{\lambda'}{\eta'} + \frac{\lambda''}{\eta''}\right), \quad [\lambda] = \langle \lambda g,_3 \rangle = \lambda' - \lambda'' \\ \{\mu\} &= \langle \mu g,_3 g,_3 \rangle = 12\left(\frac{\mu'}{\eta'} + \frac{\mu''}{\eta''}\right), \quad [\mu] = \langle \mu g,_3 \rangle = \mu' - \mu'' \end{aligned} \tag{16}$$

In this place of the note a certain two simple facts should be observed. Firstly, for the most of the tolerance effective modulus C given by (12) satisfies condition

$$\langle C \rangle - C^{eff} = [D][E]/\{F\} \tag{17}$$

for certain material constants D', D'', E', E'', F', F'' related to constituents of the laminated composite under consideration. For a certain modulus from (12) condition (17) can be reduced to

$$\langle C \rangle - C^{eff} = [D]^2/\{F\} \tag{18}$$

This universal formula (18) cannot be applied exclusively to modulus $C^{eff}_{3311} = C^{eff}_{2211}$. Indeed, for $C = C_{3311} = C^{eff}_{2211} = \lambda$ it is enough to take $F = \lambda + 2\mu$, $D = \lambda + 2\mu$ and $E = \lambda$ in (17) and then

$$\langle C \rangle - C^{o} = [\lambda + 2\mu][\lambda]/\{\lambda + 2\mu\} \tag{19}$$

Formula (19) is related to formula $(12)_3$. For the other modulus without $C_{2323} = \mu$ it is enough to take $C = C_{1111} = \lambda + 2\mu$ and $D = F = \lambda + 2\mu$ in (18) and then

$$\langle C \rangle - C^{o} = [\lambda + 2\mu]^2/\{\lambda + 2\mu\} \tag{20}$$

Formula (20) is related to formula $(12)_1$. For $C = C_{2222} = C_{3333} = \lambda + 2\mu$ it is enough to take $F = \lambda + 2\mu$ and

$$\langle C \rangle - C^{o} = [\lambda]^2/\{\lambda + 2\mu\} \tag{21}$$

Formula (21) is related to formula $(12)_2$. For $C = C_{1212} = C_{3131} = \mu$ it is enough to take $D = F = \mu$ and

$$\langle C \rangle - C^{o} = [\mu]^2/\{\mu\} \tag{22}$$

Formula (22) is related to formula $(12)_4$. For $C = C_{2233} = \lambda$ we have $D = \lambda$, $F = \lambda + 2\mu$ and

$$\langle C\rangle - C^{o} = [\lambda]^2 / \{\lambda + 2\mu\} \tag{23}$$

Formula (23) is related to formula $(12)_5$.

Now we are to shortly describe the homogenized of linear elastic two-constituent laminated microperiodic composite.

Homogenized model

This model is represented by the equation, cf. [1]

$$\langle\rho\rangle\ddot{U}_i = A^{eff}_{ijkl} U_{l,kj} = 0 \tag{24}$$

where A^{eff}_{ijkl} is the effective homogenized tensor of elastic moduli. This tensor can be introduced by applying different approaches from which the method named *homogenization of periodic tensors*, Jikov et al., [2], seems to be a suitable to realize the aim of this note. In the framework of this approach tensor A^{eff}_{ijkl} should be obtained as a result of homogenization of the tensor

$$A_{ijkl}(x_3) = A^{(1)}_{ijkl}\chi^{(1)}(x_3) + \ldots + A^{(M)}_{ijkl}\chi^{(M)}(x_3) \tag{25}$$

where $A^{(1)}_{ijkl}, \ldots, A^{(M)}_{ijkl}$ are elastic modulus related to every constituent of the laminated medium and $\chi^{(1)}(x_3), \ldots, \chi^{(M)}(x_3)$ are characteristic functions of M regions occupied by the constituents. It is mean that

$$4\xi_{ij} C^{hom}{}_{ijkl}\xi_{kl} = \inf_{u_i \in H^1(-l/2, l/2)} \langle (2\xi_{ij} + u_{i,j} + u_{j,i}) C^{hom}{}_{ijkl} (2\xi_{kl} + u_{k,l} + u_{l,k})\rangle \tag{26}$$

where $H^1(-l/2, l/2)$ is a certain Sobolev space. After the rather complicated calculation of the aforementioned infimum formula in the isotropic case of elastic modulus of laminate constituents:

$$\begin{aligned} C'_{ijkl} &= \lambda'\delta_{ij}\delta_{kl} + \mu'(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj}) \\ C''_{ijkl} &= \lambda''\delta_{ij}\delta_{kl} + \mu''(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj}) \end{aligned} \tag{27}$$

one can obtain the following open form of formula (26), cf. [2], p. 378:

$$\begin{aligned} 2\boldsymbol{\xi} \cdot \mathrm{A}^{o}\boldsymbol{\xi} = {} & \xi_{11}^2 (2\langle\mu\rangle - 4\langle(\mu - \overline{\mu})/(\lambda + 2\mu)\rangle) - \\ & - 4\xi_{11} tr\boldsymbol{\xi} \langle(\mu - \overline{\mu})(\lambda - \overline{\lambda})/(\lambda + 2\mu)\rangle + \\ & + (tr\boldsymbol{\xi})^2 (\langle(\lambda - \overline{\lambda})^2/(\lambda + 2\mu)\rangle) + \\ & + 4\langle\mu^{-1}\rangle^{-1} \sum_{i=2}^{m} \xi_{1i}^2 + 2\langle\mu\rangle \sum_{i,j=2}^{m} \xi_{ij}\xi_{ij} \end{aligned} \tag{28}$$

and effective homogenized elastic moduli in the form

$$
\begin{aligned}
&A^{\mathrm{o}}_{1111} = \langle \lambda + 2\mu \rangle - \langle (\lambda + 2\mu - \bar{\lambda} + 2\bar{\mu})^2 / (\lambda + 2\mu) \rangle = \langle (\lambda + 2\mu)^{-1} \rangle^{-1} \\
&A^{\mathrm{o}}_{2222} = A^{\mathrm{o}}_{3333} = \langle \lambda + 2\mu \rangle - \langle (\lambda - \bar{\lambda})^2 / (\lambda + 2\mu) \rangle \\
&A^{\mathrm{o}}_{1122} = A^{\mathrm{o}}_{3311} = \langle \lambda \rangle - \langle (\lambda + \mu - (\bar{\lambda} + \bar{\mu}))^2 / (\lambda + 2\mu) \rangle + \langle (\mu - \bar{\mu})^2 / (\lambda + 2\mu) \rangle \\
&A^{\mathrm{o}}_{1212} = A^{\mathrm{o}}_{3131} = \langle \mu \rangle - \langle (\mu - \bar{\mu})^2 / \mu \rangle = \langle \mu^{-1} \rangle^{-1} \\
&A^{\mathrm{o}}_{2233} = \langle \lambda \rangle - \langle (\lambda - \bar{\lambda})^2 / (\lambda + 2\mu) \rangle, \\
&A^{\mathrm{o}}_{2323} = \langle \mu \rangle
\end{aligned}
\tag{29}
$$

where symbol $\bar{x}$ for $x = \lambda, \mu$ is defined by

$$
\bar{x} \equiv \frac{\langle x / \theta \rangle}{\langle 1 / \theta \rangle} = \left(\eta' \frac{x'}{\theta'} + \eta'' \frac{x''}{\theta''} \right) \left(\eta' \frac{1}{\theta'} + \eta'' \frac{1}{\theta''} \right)^{-1} \tag{30}
$$

for θ' and θ'' taking values equal to a related material constants for every material constituent, respectively. In the framework of this note we have one from four following cases: 1) $\theta' = \lambda'$, $\theta'' = \lambda''$, or 2) $\theta' = \lambda' + 2\mu'$, $\theta'' = \lambda'' + 2\mu''$ or 3) $\theta' = \lambda' + \mu'$, $\theta'' = \lambda'' + \mu''$ or 4) $\theta' = \mu'$, $\theta'' = \mu''$.

It must be emphasized that equations (7) and (24) are identical from mathematical viewpoint but effective modulus coefficients are based on the different physical approaches. In the literature there is known many examples of periodically microstructured solids for which homogenized and tolerance effective modulus are different from mathematical viewpoint and approximately equal for a certain material properties. There also exist situations in which these modulae are identical. Hence, the question: does any clear mathematical interrelation between homogenized and tolerance effective modulus exist, seems to be fundamental. In the next paper we are going to show that in the case of two-constituent laminated periodic composite related effective modulus in the tolerance and homogenized case are simply connected under the probabilistic interpretation of second of them.

2. Probabilistic analogy; comparison of effective modulus

Let us note that for most of homogenized effective modulus (29) the related difference between averaged value $\langle C \rangle$ of modulus C and its homogenized effective value A^{o} has the form

$$
\begin{aligned}
&\langle C \rangle - A^{\mathrm{o}} = \langle (x - \bar{x})(y - \bar{y}) / \theta \rangle = \frac{\nu'}{\theta'} \big((1 - \nu') x' - \nu'' x'' \big) \big((1 - \nu') y' - \nu'' y'' \big) + \\
&+ \frac{\nu''}{\theta''} \big(\nu' x' - (1 - \nu'') x'' \big) \big(\nu' y' - (1 - \nu'') y'' \big)
\end{aligned}
\tag{31}
$$

where x', x'', y', y'' and θ', θ'', take values from all material constant of constituents and for $\nu' = \eta'$, $\nu'' = \eta''$. We are to formulate exact meaning of the formula (31) for every effective modulus. To this end a few cases will be examined.

First case deals with situations in which we have $x' = y'$ and $x'' = y''$. In this case formula (31) takes the form

$$\langle C\rangle - A^{o} = \langle (x-\overline{x})^2/\theta\rangle = \frac{\nu'}{\theta'}\left((1-\nu')x' - \nu''x''\right)^2 + \frac{\nu''}{\theta''}\left(\nu'x' - (1-\nu'')x''\right)^2 \tag{32}$$

The above special case (32) of formula (31) deals exclusively with one modulus $C = \lambda$.

Indeed, for $x = C_{1122} = C_{3311} = \lambda$ it is enough to take $\theta = \lambda + 2\mu$, $x = \lambda + 2\mu$ and $y = \lambda$ in formula (31) and arrive at

$$\begin{aligned}\langle C\rangle - A^{o} &= \langle (\lambda + 2\mu - \overline{\lambda} - 2\overline{\mu})(\lambda - \overline{\lambda})(\mu - \overline{\mu})/(\lambda + 2\mu)\rangle = \\ &= \langle (\lambda + \mu - \overline{\lambda} - \overline{\mu})^2/(\lambda + 2\mu)\rangle - \langle (\mu - \overline{\mu})^2/(\lambda + 2\mu)\rangle\end{aligned} \tag{33}$$

The above formula coincides with $(29)_3$.

Second case is related to all modulus (29) without $C = \lambda$, and $C = \mu$. Taking account formula (32) for $x = \lambda + 2\mu$ and $\theta = \lambda + 2\mu$ we obtain

$$\langle C\rangle - A^{o} = \langle (\lambda + 2\mu - \overline{\lambda} + 2\overline{\mu})^2/(\lambda + 2\mu)\rangle \tag{34}$$

The above formula coincides with $(29)_1$. For $x = \lambda$ and $\theta = \lambda + 2\mu$ we obtain

$$\langle C\rangle - A^{o} = \langle (\lambda - \overline{\lambda})^2/(\lambda + 2\mu)\rangle \tag{35}$$

The above formula coincides with $(29)_2$. For $x = \mu$ and $\theta = \mu$ we obtain

$$\langle C\rangle - A^{o} = \langle (\mu - \overline{\mu})^2/\mu\rangle \tag{36}$$

The above formula coincides with $(29)_4$. For $x = C_{2233} = \lambda$ and $\theta = \lambda + 2\mu$ we obtain

$$\langle C\rangle - A^{o} = \langle (\lambda - \overline{\lambda})^2/(\lambda + 2\mu)\rangle \tag{37}$$

The above formula coincide with $(29)_5$.

Formulas (31) and (32) together with (33)-(37) are fundamental for formulate the probabilistic analogy of homogenized effective modulus. This analogy will be understand as the method of the proof of the following lemma:

Lemma. *Tolerance and homogenized effective modulus identical for the two-constituent microperiodic laminates provided that tolerance averaged model is based on the exclusively one saw-like shape function.*
Proof of the above lemma will be decomposed onto three steps. The first step includes simple results from the probability theory [3].

***Step* 1**. *Basic tools from probabilistic theory*.

Let us consider vector-valued random variable with exclusively two known values (X_1, Y_1), (X_2, Y_2) $\in R^2$ taken with probabilities p_1 and p_2, respectively. Hence conditions $p_1 + p_2 = 1$ and $0 < p_1, p_2 < 1$ hold. Probabilistic analogy is based on the interpretation of the mentioned above random variable as a special case of two-dimensional random variable $Z = (X,Y)$ defined by

$$P(X_1,Y_1) = p_1, \quad P(X_2,Y_2) = p_2, \quad P(X_2,Y_1) = 0, \quad P(X_1,Y_2) = 0 \qquad (38)$$

Hence boundary mean values of Z are equal to:

$$\begin{aligned} E(X) &= p_1 X_1 + p_2 X_2 \\ E(Y) &= p_1 Y_1 + p_2 Y_2 \end{aligned} \qquad (39)$$

and covariance coefficient

$$\text{cov}(X,Y) = E[(X - EX)(Y - EY)] = p_1 p_2 [X][Y] \qquad (40)$$

where $[X] = X'' - X'$ i $[Y] = Y'' - Y'$. Moreover

$$D^2 X = \text{cov}(X,X) = E[(X - EX)(X - EX)] = p_1 p_2 [X]^2 \qquad (41)$$

Formulas (39), (40), (41) are started point for the subsequent steps of the proof.

***Step* 2**. *Probabilistic analogy formulation*

Let θ' and θ'' be the known positive parameters which should be interpreted as a certain constituent elastic isotropic modulae. This interpretation have been described in the previous section. Moreover denote

$$v'_\theta \equiv \frac{v'}{\theta'} / (\frac{v'}{\theta'} + \frac{v''}{\theta''}), \quad v''_\theta \equiv \frac{v''}{\theta''} / (\frac{v'}{\theta'} + \frac{v''}{\theta''}) \qquad (42)$$

where $v' = l'/l$ and $v'' = l''/l$. Hence

$$v'_\theta + v''_\theta = 1, \quad v'_\theta, v''_\theta > 0 \qquad (43)$$

and positive constants v'_θ, v''_θ can be treated as a certain probabilities p_1, p_2 related to pairs (X_1, Y_1), (X_2, Y_2) taken as the values of variable Z defined in *Step* 1. Let us assume that in the subsequent considerations probabilities p_1, p_2 take values

$$p_1 := v'_\theta, \quad p_2 := v''_\theta \tag{44}$$

Moreover, let $E_\theta X$ and $E_\theta Y$ together with $\mathrm{cov}_\theta\,(X,Y)$ and $D_\theta^2 X$ denote scalar parameters considered random variable defined by (39), (40) and (41) for p_1, p_2 defined by (44). Hence, after simple calculations one can obtain:

$$\begin{aligned} E_\theta(X) &= p_1X_1 + p_2X_2 = (\frac{v'}{\theta'}X_1 + \frac{v''}{\theta''}X_2)/(\frac{v'}{\theta'} + \frac{v''}{\theta''}) \\ E_\theta(Y) &= p_1Y_1 + p_2Y_2 = (\frac{v'}{\theta'}Y_1 + \frac{v''}{\theta''}Y_2)/(\frac{v'}{\theta'} + \frac{v''}{\theta''}) \end{aligned} \tag{45}$$

and

$$\begin{aligned} \mathrm{cov}_\theta(X,Y) &= E_\theta[(X - E_\theta X)(Y - E_\theta Y)] = \\ &= v'_\theta v''_\theta[X][Y] = \frac{\theta'\theta''v'v''}{(\theta''v' + \theta'v'')^2}[X][Y] \end{aligned} \tag{46}$$

Moreover

$$\begin{aligned} D_\theta^{\,2}(X,Y) &= E_\theta[(X - E_\theta X)]^2 = \\ &= v'_\theta v''_\theta[X]^2 = \frac{\theta'\theta''v'v''}{(\theta''v' + \theta'v'')^2}[X]^2 \end{aligned} \tag{47}$$

Now we are to realize the crucial part of the proof.

***Step* 3**. *Crucial part of the proof*

To finish the proof we should repeat observations from the last section. The first from this observations is that covariants of the form (17) or their special case (18) represent tolerance effective modulus. The second from this observations is that covariants of the form (31) or their special case (32) represent homogenized effective modulus. It means that if we prove that differences of the form (17) connected to tolerance effective modulus are equal to the differences of the form (31) connected to the related homogenized effective modulus then we can conclude that tolerance and homogenized effective modulus tensors are identical. We are to show that the aforementioned differences are equal.

To this end let us take into account (46). By virtue of (13) we have

$$\{\theta\} = \frac{\theta'}{v'} + \frac{\theta''}{v''} \tag{48}$$

and then

$$\frac{[x][y]}{\{\theta\}} = \frac{[x][y]}{\frac{\theta'}{v'} + \frac{\theta''}{v''}} = \frac{v'v''[x][y]}{v''\theta' + v'\theta''} \tag{49}$$

From the other hand side, since

$$\langle 1/\theta \rangle = \frac{v'}{\theta'} + \frac{v''}{\theta''} \tag{50}$$

and

$$\overline{x} \equiv \langle x/\theta \rangle / \langle 1/\theta \rangle = v'_\theta x' + v''_\theta x'' \equiv E_\theta x \tag{51}$$

we have

$$\langle (x-\overline{x})(y-\overline{y})/\theta \rangle = \langle 1/\theta \rangle E_\theta (x - E_\theta x)(y - E_\theta y) = v'_\theta v''_\theta [x][y] \tag{52}$$

together with a special case of (52).

$$\langle (x-\overline{x})^2/\theta \rangle = \langle 1/\theta \rangle E_\theta (x - E_\theta x) = v'_\theta v''_\theta [x]^2 \tag{53}$$

By comparison of (49) and (52) by virtue of (46) one can obtain

$$\langle (x-\overline{x})(y-\overline{y})/\theta \rangle = \frac{[x][y]}{\{\theta\}} \tag{54}$$

Just proved relation (54) means that tolerance and homogenized effective modulus tensors are identical. This ends the proof.

3. Final remarks

It was shown that, in the case in which constituents are isotropic, effective modulae are identical, provided that we deal with laminated composite with microperiodic structure. It a well known fact that there exist composites with microperiodic structure for which tolerance (for a certain choice of a number and a form shape functions) and homogenized modulus are different. For the most cases an answer to questions: *Under what assumptions tolerance and homogenized modulus are identical?* and *Have mentioned above probability analogy ever plays any significant role in the connection within these modulus*? are still open.

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