FREE VIBRATION OF ANNULAR AND CIRCULAR PLATES OF STEPPED THICKNESS. APPLICATION OF GREEN'S FUNCTION METHOD

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Abstract. The paper concerns axisymmetric free vibration of annular and circular plates of stepped thickness with elastic ring supports. Exact solution to the problem was obtained by dividing of considered plate into annular plates of uniform thickness and by using Green's function method. Analytical solution to vibration problem was used to perform numerical frequency analysis of an exemplary stepped annular plate.

Introduction

The vibrations of annular and circular plates of stepped thickness have been studied by several authors (for instance references [1-3]). The solutions to free vibration problems concerning such plates were obtained often by using an approximate method. In references [1, 2] finite element method and optimized Rayleigh-Ritz method were applied. A closed form of exact solution to the problem can be obtained by using the Green's function method. This method in the previous papers [3-6] was used, but the problems there considered concern the axisymmetric free vibrations of annular or circular uniform plates with elastic ring supports.

The present paper deals with free vibrations of annular and circular plates of stepped thickness with elastic ring supports. Exact solution to considered vibration problem is obtained by using properties of Green's functions corresponding to differential operators which occur in the mathematical description of the plate vibration. The Green's functions are derived by solving auxiliary problems. The next step in this approach consists in dividing of the stepped plate into uniform plate elements: annular plates or one circular plate and annular plates. In formulation and solution of the problem one takes into account arbitrary finite number of the plate elements. Analytical solution to vibration problem was used to perform numerical analysis of influence of parameters characterizing the system on its eigenfrequencies. In the analysis free vibration of stepped annular and circular plates composed of two or more uniform plate elements were considered.

1. Formulation and solution to the problem

Consider an annular or circular plate of thickness varying stepwise along $(n - 1)$ concentric circles as schematic shown in Figure 1. These circles mark out n plate elements - uniform annular plates of thickness h_j and radii a_{j-1}, a_j, where $a_{j-1} < a_j$ ($j = 1,\ldots,n$). In case of circular plate the inner element is a circular plate of radius a_1 and other elements similarly as for annular plate are appointed.

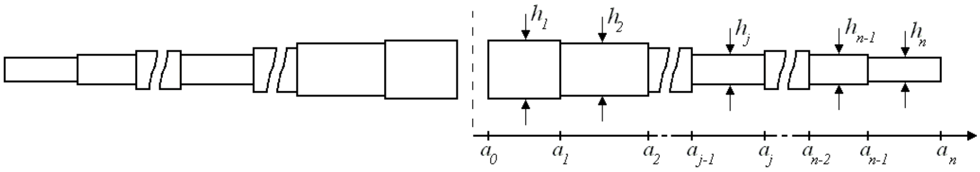

Fig. 1. Cross-section of stepped annular plate divided into n uniform plates

Axisymmetric vibration of j-th plate element is governed by differential equation:

$$D_j \frac{1}{r}\frac{\partial}{\partial r}\left\{ r \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial w_j}{\partial r}\right)\right]\right\} + \bar{\rho}_j h_j \frac{\partial^2 w_j}{\partial t^2} = \tag{1}$$
$$= q_j - s_{j-1}\,\delta\left(r - a_{j-1}\right) + m_{j-1}\,\delta'\left(r - a_{j-1}\right) + s_j\,\delta\left(r - a_j\right) - m_j\,\delta'\left(r - a_j\right)$$

where $w_j = w_j(r, t)$ is transverse displacement of the plate, r, t - radial and time variable, $D_j = E\,h_j^3 / 12\,(1 - \nu^2)$ - bending rigidity of the plate, E - Young modulus, ν - Poisson ratio, $q_j = q_j(r, t)$ - load per unit area, $\bar{\rho}_j$ - mass per unit volume, h_j - thickness of the j-th uniform plate element, $s_j = s_j(t)$ is the shearing force, $m_j = m_j(t)$ is the bending moment and $\delta(\ )$ is the Dirac delta function. One assumes that: $s_0 = m_0 = s_n = m_n = 0$.

Equation (1) is completed by boundary and continuity conditions. The boundary conditions corresponding to annular or circular plates with clamped or simply supported or free edges can be written symbolically in the form:

$$\mathrm{B}_0\left[w_0\right]\big|_{r=a_0} = 0, \qquad \mathrm{B}_n\left[w_n\right]\big|_{r=a_n} = 0 \tag{2}$$

for annular plates and

$$\mathrm{B}_n\left[w_n\right]\big|_{r=a_n} = 0 \tag{2'}$$

for circular plates. The continuity conditions are:

$$w_j\left(a_j, t\right) = w_{j+1}\left(a_j, t\right), \qquad w'_j\left(a_j, t\right) = w'_{j+1}\left(a_j, t\right),\ j = 1,\ldots,n-1 \tag{3}$$

where "prim" denotes differentiation with respect to r.

We assume additionally that the considered plate is supported on elastic concentric rings. Let N_j denote the number of supporting rings for j-th plate element and r_{jl} - radii of supporting rings ($l = 1,\ldots,N_j$, $j = 1,\ldots,n$). Then the function q_j occurring in equation (1), has a form:

$$q_j(r,t) = -\sum_{l=1}^{N_j} k_{jl}\, w(r,t)\,\delta(r - r_{jl}) \tag{4}$$

where k_{jl} denotes the stiffness coefficient of the l-th supporting ring which occurs in range of the j-th plate element.

In case of free harmonic vibration of the system with eigenfrequency ω, for each plate element of constant thickness, one assumes that

$$w_j(r,t) = W_j(r)\, e^{i\omega t}, \qquad s_j(t) = S_j\, e^{i\omega t}, \qquad m_j(t) = M_j\, e^{i\omega t} \tag{5}$$

Taking into account (5) and introducing dimensionless quantities: $\bar{r}_j = r / a_j$, $\bar{r}_{jl} = r_{jl} / a_j$ and $\overline{W}_j = W_j / a_j$, in equation (1) and using continuity conditions (3), one obtains

$$\begin{aligned}\mathbf{L}_j\left[\overline{W}_j\right] = &-\sum_{l=1}^{N_j} K_{jl}\, \overline{W}_j\, \bar{r}_j \delta(\bar{r}_j - \bar{r}_{jl}) \\ &- \overline{S}_{j-1} \frac{\varDelta_j}{\alpha_j^{\,2}} \bar{r}_j\, \delta(\bar{r}_j - \alpha_j) + \overline{M}_{j-1} \frac{\varDelta_j}{\alpha_j} \bar{r}_j\, \delta'(\bar{r}_j - \alpha_j) + \overline{S}_j\, \bar{r}_j\, \delta(\bar{r}_j - 1) - \overline{M}_j\, \bar{r}_j\, \delta'(\bar{r}_j - 1)\end{aligned} \tag{6}$$

$$\overline{W}_j(1) = \frac{1}{\alpha_{j+1}} \overline{W}_{j+1}(\alpha_{j+1}) \quad \overline{W}_j^{\,'}(1) = \overline{W}_{j+1}^{\,'}(\alpha_{j+1}),\ j = 1, \ldots, n-1 \tag{7}$$

where: $\mathbf{L}_j\left[\overline{W}_j\right] \equiv \dfrac{\mathrm{d}}{\mathrm{d}\bar{r}_j}\left\{\bar{r}_j \dfrac{\mathrm{d}}{\mathrm{d}\bar{r}_j}\left[\dfrac{1}{\bar{r}_j}\dfrac{\mathrm{d}}{\mathrm{d}\bar{r}_j}\left(\bar{r}_j \dfrac{\mathrm{d}\overline{W}_j}{\mathrm{d}\bar{r}_j}\right)\right]\right\} - \bar{r}_j \varOmega_j^{\,4} \overline{W}_j$, $\quad \varOmega_j^{\,4} = \dfrac{\bar{\rho}_j h_j \omega^2 a_j^4}{D_j}$,

$K_{jl} = k_{jl}\, a_j^3 / D_j$, $\quad \alpha_j = a_{j-1} / a_j$, $\quad \varDelta_j = D_{j-1} / D_j$, $\quad \overline{S}_j = S_j\, a_j^2 / D_j$, $\overline{M}_j = M_j\, a_j / D_j$, $j = 1,\ldots,n$.

Solution of equation (6) is obtained by using Green's function method. This method consist in determine of an integral operator - an inverse to the differential operator which occurs in the considered differential problem. The integral operator is defined by a function (Green's function), which first should be determined. Green's function of an operator is obtained as a solution of an auxiliary problem. Determination of Green's function of the operator $\mathbf{L}_j$ occurring in equation (6) is presented in the next chapter of this paper.

Ones the Green's function $G^j(r, \rho)$ of the operator $\mathbf{L}_j$ is known then the solution of equation (6) can be presented in form [3]:

$$W_j(r_j) = -\sum_{l=1}^{N_j} K_{jl}\, W_j(r_{jl})\, r_{jl}\, G^j(r_j, r_{jl}) - \tilde{S}_{j-1} \frac{\Delta_j}{\alpha_j} G^j(r_j, \alpha_j; \Omega_j)$$
$$- \tilde{M}_{j-1}\, \Delta_j G^j_{,\rho}(r_j, \alpha_j; \Omega_j) + \tilde{S}_j G^j(r_j, 1; \Omega_j) + \tilde{M}_j G^j_{,\rho}(r_j, 1; \Omega_j) \qquad (8)$$

where the dashes over symbols r, W, S, M are omitted for clarity.

In order to obtain the characteristic equation of the considered vibration problem, a set of homogeneous equations with respect to unknown quantities $\tilde{S}_j$, $\tilde{M}_j$ (j = 1,...,n – 1) and $W_j(r_{jl})$ (l = 1,…,N_j, j = 1,…,n) is created.

The 2 (n – 1) linear equations of the set are obtained by taking into account equation (8) in continuity conditions (7). Another equations are obtained by substituting $r_j = r_{jl}$ in equation (8). The system of equations can be written in a matrix form as follows:

$$\boldsymbol{A}\,\boldsymbol{X} = \mathbf{0} \qquad (9)$$

Where $\boldsymbol{X} = \left[\tilde{S}_1, \tilde{M}_1, ..., \tilde{S}_{n-1}, \tilde{M}_{n-1}, W_1(r_{11}), ..., W_n(r_{nN_n})\right]^T$. Equation (9) has non-trivial solution if and only if

$$\det \boldsymbol{A}(\omega) = 0 \qquad (10)$$

Equation (10) is the characteristic equation to the problem. This equation is than solved numerically with respect to natural frequency ω.

2. Green's functions

Green's function $G(r, \rho)$ of the operator $\mathbf{L}$ considered in this study is a solution of equation

$$\mathbf{L}[G(r, \rho)] = \delta(r - \delta) \qquad (11)$$

where $\mathbf{L}[G] \equiv \frac{\mathrm{d}}{\mathrm{d}r}\left\{ r \frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r}\left(r \frac{\mathrm{d}G}{\mathrm{d}r} \right)\right]\right\} - r\,\Omega^4 G$ and $\Omega^4 = \frac{\bar{\rho}\, h \omega^2 a^4}{D}$. Green's function G corresponding to uniform annular plate with both edges ($r = b$, $r = a$, $b < a$) simply supported (S – S plate) satisfies following boundary conditions:

$$G\big|_{r=\alpha,1} = 0 \qquad \left.\left(\frac{\mathrm{d}^2 G}{\mathrm{d}r^2} + \nu \frac{1}{r} \frac{\mathrm{d}G}{\mathrm{d}r} \right)\right|_{r=\alpha,1} = 0 \qquad (12)$$

where $\alpha = b/a$. Green's functions corresponding to annular plates with the other boundary conditions were considered in references [3, 4, 6].

Solution of equation (11) can be presented in a form [5]:

$$G(r,\rho) = G_0(r,\rho) + G_1(r,\rho) H(r-\rho) \tag{13}$$

where $G_0(r, \rho)$ is a general solution of homogeneous equation

$$\mathbf{L}[G(r,\rho)] = 0 \tag{14}$$

and $G_1(r, \rho)\, H(r-\rho)$ is a particular solution of equation (11). It may be shown that G_1 is a solution of equation (14) which satisfies following conditions:

$$G_1\big|_{r=\rho} = \left.\frac{\partial G_1}{\partial r}\right|_{r=\rho} = \left.\frac{\partial^2 G_1}{\partial r^2}\right|_{r=\rho} = 0 \text{ and } \left.\frac{\partial^3 G_1}{\partial r^3}\right|_{r=\rho} = \frac{1}{\rho} \tag{15}$$

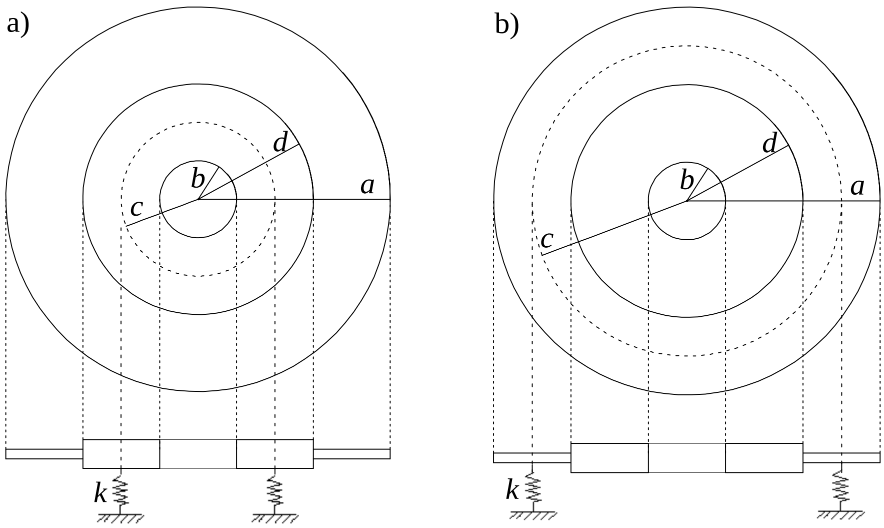

Fig. 2. Stepped annular plates with one ring support

The general solution of homogeneous equation (14) can be expressed in terms of Bessel functions J_0, Y_0 and modified Bessel functions I_0, K_0, of the first and second kind. Function G_1 viz. has the form

$$G_1(r,\rho) = c_1 J_0(r\Omega) + c_2 I_0(r\Omega) + c_3 Y_0(r\Omega) + c_4 K_0(r\Omega) \tag{16}$$

where c_i's are integral constants. The constants are determined by using conditions (15). Solving a system of equations obtained on the basis of (15) and using the following relationships [7]:

$$I_0(z)K_1(z)+I_1(z)K_0(z)=\frac{1}{z}$$

$$Y_0(z)\,J_1(z)-Y_1(z)\,J_0(z)=\frac{2}{\pi\,z}$$

function G_1 can be written as

$$G_1(r,\rho)=\frac{1}{2\Omega^2}\Big(I_0(r\,\Omega)K_0(\rho\,\Omega)-I_0(\rho\,\Omega)K_0(r\,\Omega)+ \\ +\frac{\pi}{2}\big(J_0(r\,\Omega)Y_0(\rho\,\Omega)-J_0(\rho\,\Omega)Y_0(r\,\Omega)\big)\Big) \tag{17}$$

Function G_0 as a general solution of homogeneous equation (14) has the form analogous to function G_1 given by equation (16):

$$G_0(r,\rho)=C_1J_0(r\,\Omega)+C_2I_0(r\,\Omega)+C_3Y_0(r\,\Omega)+C_4K_0(r\,\Omega) \tag{18}$$

Table 1

Values of frequency parameter $\boldsymbol{\Omega_1=\sqrt{\rho h_1/D_1}\,\omega_1 a^2}$ obtained by present method and by Rayleigh-Ritz method (italic, reference [2]) for free-free stepped annular plate simply supported along a circle with radius *c*

		h_2/h_1 = 0.6			h_2/h_1 = 0.8		
		d/*a*			*d*/*a*		
b/*a*	*c*/*a*	0.3	0.5	0.7	0.4	0.6	0.8
0.1	0.2	3.213	4.117	4.542	3.863	4.141	4.176
		3.29	*4.14*	*4.55*	*3.86*	*4.15*	*4.18*
	0.4	3.880	5.200	6.268	5.066	5.644	5.803
			5.27	*6.30*		*5.66*	*5.81*
0.4	0.5		3.972	5.217		4.686	5.037
				5.21		*4.68*	*5.04*
	0.6		5.152	6.905		6.234	6.817
				6.94			*6.82*

The constants C_i‘s are determined by taking into account equations (17), (18) in (13) and by using boundary conditions (12). The following functions are introduced to present the function G_0:

$$\Phi_{S1}(z)=2\,z\,J_0(z)I_0(z)-(1-\nu)\big(J_1(z)I_0(z)+J_0(z)I_1(z)\big)$$

$$\Phi_{S2}(z)=2\,z\,J_0(z)K_0(z)-(1-\nu)\big(J_1(z)K_0(z)-J_0(z)K_1(z)\big)$$

$$\Phi_{S3}(z) = 2\,z\,Y_0(z)I_0(z) - (1-\nu)(Y_1(z)I_0(z) + Y_0(z)I_1(z))$$

$$\Phi_{S4}(z) = 2\,z\,Y_0(z)K_0(z) - (1-\nu)(Y_1(z)K_0(z) - Y_0(z)K_1(z))$$

$$\Psi_{S1}(z,u) = \frac{1-\nu}{z} J_0(u) + K_0(u)\Phi_{S1}(z) - I_0(u)\Phi_{S2}(z)$$

$$\Psi_{S2}(z,u) = \frac{1-\nu}{z} Y_0(u) + K_0(u)\Phi_{S3}(z) - I_0(u)\Phi_{S4}(z)$$

$$\Psi_{S3}(z,u) = \frac{1-\nu}{z} I_0(u) + \frac{\pi}{2}(Y_0(u)\Phi_{S1}(z) - J_0(u)\Phi_{S3}(z))$$

$$\Psi_{S4}(z,u) = \frac{1-\nu}{z} K_0(u) + \frac{\pi}{2}(Y_0(u)\Phi_{S2}(z) - J_0(u)\Phi_{S4}(z))$$

and

$$d = \frac{4(1-\nu)^2}{\alpha\,\pi\,\Omega^2} + \\ + \Phi_{S1}(\Omega)\Phi_{S4}(\alpha\,\Omega) + \Phi_{S1}(\Omega)\Phi_{S4}(\alpha\,\Omega) - \Phi_{S2}(\Omega)\Phi_{S3}(\alpha\,\Omega) - \Phi_{S3}(\Omega)\Phi_{S2}(\alpha\,\Omega)$$

The function G_0 (for S – S plate) can be written using above functions as follows:

$$G_0(r,\rho) = \frac{1}{2\,d\,\Omega^2}(\Psi_{S1}(\Omega,\rho\,\Omega)\Psi_{S2}(\alpha\,\Omega,r\,\Omega) - \Psi_{S2}(\Omega,\rho\,\Omega)\Psi_{S1}(\alpha\,\Omega,r\,\Omega) + \\ + \frac{2}{\pi}(\Psi_{S3}(\Omega,\rho\,\Omega)\Psi_{S4}(\alpha\,\Omega,r\,\Omega) - \Psi_{S4}(\Omega,\rho\,\Omega)\Psi_{S3}(\alpha\,\Omega,r\,\Omega))) \tag{19}$$

Finally, Green's function G of operator L corresponding to simply supported annular plate is given by taking (17) and (19) in (13). Green's functions of operators L corresponding to circular and annular plates with other boundary conditions are presented in papers [3-6].

3. Numerical examples

Numerical example concerns the free vibration of a stepped annular plate with both edges free (*r* = *a*, *r* = *b*) and one elastic ring support. Let *c* and *k* denote radius and stiffness coefficient of the supporting ring, respectively, ($b \le c \le a$). Plate thickness varies stepwise in radial direction along the circle of radius *d* ($b < d < a$). The plate is divided into two plate elements - free-free annular plates of uniform thickness (Fig. 2). Figure 3 presents values of frequency parameter Ω_1 as a function of ratio *c*/*a*, for various values of stiffness parameter $K = k\,a^3 / D_2$. We

can see both parameters: ratio c/a and stiffness parameter K, have significant effect on frequency parameter Ω_1. If $k \to \infty$ then the frequency parameter of free-free stepped annular plate simply supported along a circle of radius c, is obtained. Results of numerical calculations are presented in Table 1. The frequency parameters are compared (where available) with results obtained by optimized Rayleigh-
-Ritz method which are presented in reference [2]. All calculations were performed for $\nu = 0.3$.

Conclusions

Green's function method was used to obtain exact solution to free vibration problem of stepped annular and circular plates with elastic ring supports. In formulation and solution to vibration problem one takes into account an arbitrary numbers of thickness steps of the plate and elastic ring supports. The analysis concerns axisymmetric vibration of annular and circular plates with various boundary conditions.

Numerical analysis shows that the parameters characterizing the plate (thickness ratio of plate components, radii of components, locations and stiffness coefficients of ring supports) have significant effect on free vibration frequency of the plate. Results obtained by presented here method were compared with results obtained by optimized Rayleigh-Ritz method. Good agreement of results confirms the correctness of presented method.

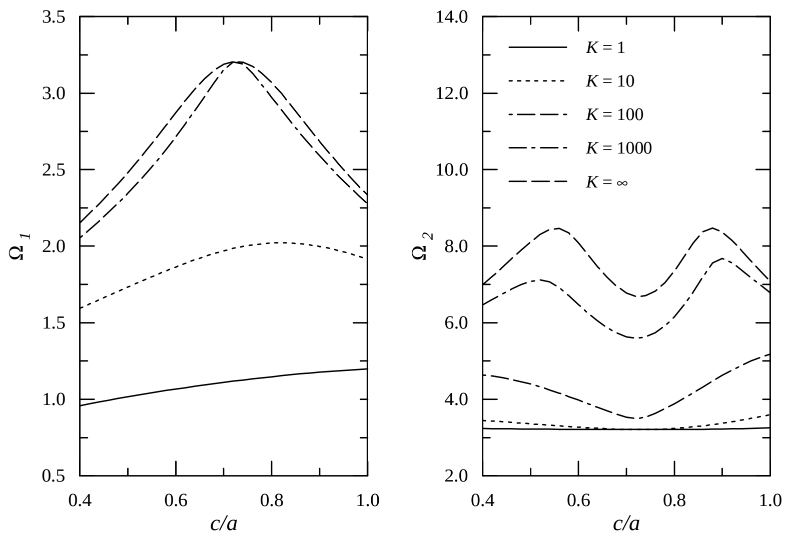

Fig. 3. Values of frequency parameter $\Omega_i = \sqrt[4]{\overline{\rho}_2\, h_2\, \omega_i^2\, a^4 / D_2}$, ($i$ = 1, 2) of free-free stepped annular plate as a function of ratio c/a, for various values of stiffness parameter $K = ka^3/D_2$; b/a = 0.4, d/a = 0.7, h_2/h_1 = 0.8

References

[1] Avalos D.R., Larrondo H.A., Sonzogni V., Laura P.A.A., A general approximate solution of the problem of free vibrations of annular plates of stepped thickness, Journal of Sound and Vibration 1996, 196, 3, 275-283.

[2] Gutierrez R.H., Laura P.A.A., Fundamental frequency of an annular circular plate of non-uniform thickness and an intermediate concentric circular support, Journal of Sound and Vibration 1999, 224, 4, 775-779.

[3] Kukla S., Szewczyk M., Analiza drgań własnych płyt pierścieniowych o skokowo zmiennej grubości, send to Zeszyty Naukowe Katedry Mechaniki Stosowanej Politechniki Śląskiej.

[4] Kukla S., Szewczyk M., Application of Green's function method in frequency analysis of axisymmetric vibration of annular plates with elastic ring supports, Scientific Research of the Institute of Mathematics and Computer Science 2005, 1, 4, 79-86.

[5] Kukla S., Szewczyk M., The Green's functions for vibration problems of circular plates with elastic ring supports, Scientific Research of the Institute of Mathematics and Computer Science 2004, 1, 3, 67-72.

[6] Kukla S., Szewczyk M., Frequency analysis of annular plates with elastic concentric supports by using a Green's function method, Journal of Sound and Vibration (in print).

[7] McLachlan N.W., Funkcje Bessela dla inżynierów, PWN, Warszawa 1964.