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ON THE TANGENCY OF THE RECTIFIABLE ARCS

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Abstract. In this paper some problems of the tangency of the rectifiable arcs in generalized metric spaces (E,l) are considered. Some sufficient and necessary conditions for the tangency of these arcs have been given here.

Introduction

Let (E,l) be a generalized metric space. E denotes here an arbitrary non-empty set, and l is a non-negative real function defined on the Cartesian product $E_0 \times E_0$ of the family E_0 of all non-empty subsets of the set E.

Let k be any, but fixed positive real number, and let a, b be arbitrary non-negative real functions defined in a certain right-hand side neighbourhood of 0 such that

$$a(r) \xrightarrow[r\to 0^+]{} 0 \quad \text{and} \quad b(r) \xrightarrow[r\to 0^+]{} 0 \tag{1}$$

We say that a pair (A,B) of sets of the family E_0 is (a,b)-clustered at the point p of the space (E,l), if 0 is the cluster point of the set of all numbers $r > 0$ such that the sets $A \cap S_l(p,r)_{a(r)}$ and $B \cap S_l(p,r)_{b(r)}$ are non-empty.

The sets $S_l(p,r)_{a(r)}$ and $S_l(p,r)_{b(r)}$ (see [13]) denote here so-called $a(r)$- and $b(r)$-neighbourhoods of the sphere $S_l(p,r)$ with the centre at the point $p \in E$ and the radius $r > 0$ in the space (E,l), respectively.

The tangency relation $T_l(a,b,k,p)$ of sets of the family E_0 in the generalized metric space (E,l) is defined as follows (see [13]):

$$\begin{aligned} T_l(a,b,k,p) = \{(A,B)\colon\ & A,B \in E_0, \text{ the pair } (A,B) \text{ is } (a,b)\text{-clustered} \\ & \text{at the point } p \text{ of the space } (E,l) \text{ and} \\ & \frac{1}{r^k} l(A \cap S_l(p,r)_{a(r)}, B \cap S_l(p,r)_{b(r)}) \xrightarrow[r\to 0^+]{} 0\} \end{aligned} \tag{2}$$

If $(A,B) \in T_l(a,b,k,p)$, then we say that the set $A \in E_0$ is (a,b)-tangent (or briefly: is tangent) of order k to the set $B \in E_0$ at the point p of the space (E,l).

Let ρ be an arbitrary metric of the set E. We shall denote by $d_\rho A$ the diameter of the set $A \in E_0$, and by $\rho(A,B)$ the distance of sets $A,B \in E_0$ in the metric space (E,ρ).

Let f be any subadditive increasing real function defined in a certain right-hand side neighbourhood of 0, such that $f(0)=0$. By $\mathfrak{F}_{f,\rho}$ we will denote the class of all functions l fulfilling the conditions:

1^0 $l:\, E_0 \times E_0 \longrightarrow \langle 0,\infty)$,

2^0 $f(\rho(A,B)) \le l(A,B) \le f(d_\rho(A\cup B)) \quad \text{for} \quad A,B \in E_0$.

It is easy to check that every function $l \in \mathfrak{F}_{f,\rho}$ generates in the set E the metric l_0 defined by the formula:

$$l_0(x,y) = l(\{x\},\{y\}) = f(\rho(x,y)) \quad \text{for} \quad x,y \in E \tag{3}$$

In this paper we shall consider certain problems concerning the tangency of the rectifiable arcs of the classes A_p and $\widetilde{A}_p$ in generalized metric spaces (E,l), where $l \in \mathfrak{F}_{f,\rho}$. Some theorems for the tangency of the arcs of these classes have been given here.

1. The tangency of the rectifiable arcs of the class $\widetilde{A}_p$

Let ρ be a metric of the set E, and let A be any set of the family E_0 of all non-empty subsets of the set E. By A' we shall denote the set of all cluster points of the set A of the family E_0.

The classes of sets $\widetilde{A}_p$, mentioned in the Introduction of this paper, is defined as follows (see papers [1, 11, 12]):

$\widetilde{A}_p = \{A \in E_0$: A is rectifiable arc with the origin at the point $p \in E$ and

$$\lim_{A\,\ni x \to p} \frac{\ell(\widetilde{px})}{\rho(p,x)} = g < \infty\} \tag{4}$$

where $\ell(\widetilde{px})$ denotes the lenght of the arc $\widetilde{px}$ with the ends p and x.

If $g=1$, then we say that the rectifiable arc $A \in E_0$ has the Archimedean property at the point p of the metric space (E,ρ), and is the arc of the class A_p.

In the paper [11] W. Waliszewski proved (see Theorem 2) that the class of arcs $\widetilde{A}_p$ is contained in the class of sets A_p^* defined by the formula:

$$A_p^* = \{A \in E_0\colon\ p \in A' \text{ and there exists a number } \lambda > 0 \text{ such that}$$
$$\limsup_{[A;p]\ni(x,y)\to(p,p)} \frac{\rho(x,y) - \lambda\rho(x,A)}{\rho(p,x)} \le 0\} \tag{5}$$

where

$$[A;p] = \{(x,y)\colon\ x \in E,\ \ y \in A \ \text{ oraz } \ \rho(x,A) < \rho(p,x) = \rho(p,y)\} \tag{6}$$

and

$$\rho(x,A) = \inf\{\rho(x,y)\colon\ y \in A\} \quad \text{for} \quad x \in E \tag{7}$$

From the considerations of the papers [1, 11, 12] it follows that the class of sets A_p^* is contained (for $k = 1$) in the class $\widetilde{M}_{p,k}$:

$$\widetilde{M}_{p,k} = \{A \in E_0\colon\ p \in A' \ \text{ and there exists } \ \mu > 0 \ \text{ such that}$$
$$\text{for an arbitrary } \ \varepsilon > 0 \ \text{ there exists } \ \delta > 0 \ \text{ such that}$$
$$\text{for every pair of points } \ (x,y) \in [A,p;\mu,k]$$
$$\text{if } \ \rho(p,x) < \delta \ \text{ and } \ \frac{\rho(x,A)}{\rho^k(p,x)} < \delta, \ \text{ then } \ \frac{\rho(x,y)}{\rho^k(p,x)} < \varepsilon\} \tag{8}$$

where

$$[A,p;\mu,k] = \{(x,y)\colon\ x \in E,\ \ y \in A \ \text{ and } \ \mu\rho(x,A) < \rho^k(p,x) = \rho^k(p,y)\}.$$

We say (see [6]) that the set $A \in E_0$ has the Darboux property at the point p of the generalized metric space (E,l), and we shall write this as: $A \in D_p(E,l)$, if there exists a number $\tau > 0$ such that $A \cap S_l(p,r) \neq \emptyset$ for $r \in (0,\tau)$.

Because any rectifiable arc A with the origin at the point $p \in E$ has the Darboux property at the point p of the generalized metric space (E,l), then from here and from the above definition of the class of sets $\widetilde{M}_{p,k}$ it follows that $\widetilde{A}_p \subset \widetilde{M}_{p,1} \cap D_p(E,l)$.

From Theorem 2.1 of the paper [10] and from the above inclusion it follows the following corollary:

Corollary 1.1. *If in the metric space (E,ρ) the arc A belongs to the class $\widetilde{A}_p$, then*

$$\frac{a(r)}{r} \xrightarrow[r\to 0^+]{} 0 \tag{9}$$

if and only if

$$\frac{1}{r} d_\rho(A \cap S_\rho(p,r)_{a(r)}) \xrightarrow[r\to 0^+]{} 0 \tag{10}$$

Using this corollary we shall prove:

Theorem 1.1. *If for arbitrary function $l \in \mathfrak{F}_{f,\rho}$ and rectifiable arcs $A, B \in \widetilde{A}_p$ the pair $(A,B) \in T_l(a,b,1,p)$ in the generalized metric space (E,l), then*

$$\frac{a(r)}{r} \xrightarrow[r\to 0^+]{} 0 \quad \textit{and} \quad \frac{b(r)}{r} \xrightarrow[r\to 0^+]{} 0 \tag{11}$$

`Proof.` We assume that $(A,B) \in T_l(a,b,1,p)$ for $A, B \in \widetilde{A}_p$ and $l \in \mathfrak{F}_{f,\rho}$. From here, putting $l = f \circ d_\rho$ we obtain

$$\frac{1}{r} f(d_\rho((A \cap S_l(p,r)_{a(r)}) \cup (B \cap S_l(p,r)_{b(r)}))) \xrightarrow[r\to 0^+]{} 0 \tag{12}$$

Because

$$d_\rho(A \cap S_l(p,r)_{a(r)}) \le d_\rho((A \cap S_l(p,r)_{a(r)}) \cup (B \cap S_l(p,r)_{b(r)})),$$

and

$$d_\rho(B \cap S_l(p,r)_{b(r)}) \le d_\rho((A \cap S_l(p,r)_{a(r)}) \cup (B \cap S_l(p,r)_{b(r)})),$$

then from here, from (12) and from the properties of the function f follows

$$\frac{1}{r} f(d_\rho(A \cap S_l(p,r)_{a(r)})) \xrightarrow[r\to 0^+]{} 0 \tag{13}$$

and

$$\frac{1}{r} f(d_\rho(B \cap S_l(p,r)_{b(r)})) \xrightarrow[r\to 0^+]{} 0 \tag{14}$$

Hence and from the equality

$$f(d_\rho A) = d_l A \quad \text{for} \quad A \in E_0 \tag{15}$$

we obtain

$$\frac{1}{r} d_l(A \cap S_l(p,r)_{a(r)}) \xrightarrow[r\to 0^+]{} 0 \tag{16}$$

and

$$\frac{1}{r} d_l(B \cap S_l(p,r)_{b(r)}) \xrightarrow[r \to 0^+]{} 0 \tag{17}$$

Because every function $l \in \mathfrak{F}_{f,\rho}$ generates in the set E the metric l_0, then from here, from (16), (17) and from Corollary 1.1 follows the thesis of this theorem.

Theorem 1.1 has fundamental meaning for the tangency of the rectifiable arcs in the generalized metric spaces (E, l). From above theorem it follows that the condition (11) is sufficient and necessary condition, among other things, for the compatibility, equivalence, additivity and homogeneity of the tangency relation $T_l(a, b, 1, p)$ of the rectifiable arcs of the class $\widetilde{A}_p$.

Below we shall prove:

Theorem 1.2. *If in the generalized metric space (E, l) the function $l \in \mathfrak{F}_{f,\rho}$, and the rectifiable arcs $A, B \in \widetilde{A}_p$ are subsets of a certain arc $C \in \widetilde{A}_p$, then $(A, B) \in T_l(a, b, 1, p)$ if and only if the functions a, b fulfil the condition* (11).

`Proof.` We assume that the functions a, b fulfil the condition (11). Let $\alpha = \max(a, b)$. Hence, from (11), from Theorem 2.1 of the paper [10] and from the fact that every function $l \in \mathfrak{F}_{f,\rho}$ generates in the set E the metric l_0 it follows that

$$\frac{1}{r} d_l(C \cap S_l(p,r)_{\alpha(r)}) \xrightarrow[r \to 0^+]{} 0 \tag{18}$$

Because from the assumptions of this theorem

$$A \cap S_l(p,r)_{a(r)} \subset C \cap S_l(p,r)_{\alpha(r)} \text{ and } B \cap S_l(p,r)_{b(r)} \subset C \cap S_l(p,r)_{\alpha(r)} \tag{19}$$

then from here and from (18) follows

$$\frac{1}{r} d_l(A \cap S_l(p,r)_{a(r)}) \xrightarrow[r \to 0^+]{} 0 \tag{20}$$

and

$$\frac{1}{r} d_l(B \cap S_l(p,r)_{b(r)}) \xrightarrow[r \to 0^+]{} 0 \tag{21}$$

Let by the definition $\rho_l(A, B)$ denote the distance of sets $A, B \in E_0$ in the generalized metric space (E, l), i.e.,

$$\rho_l(A, B) = \inf\{l_0(x, y) \colon x \in A,\ y \in B\} \quad \text{for} \quad A, B \in E_0 \tag{22}$$

From the equality (3) and from the properties of the function f follows

$$f(\rho(A,B))=f(\inf\{\rho(x,y):\ x\in A,\ y\in B\})$$

$$=\inf\{f(\rho(x,y)):\ x\in A,\ y\in B\}=\inf\{l_0(x,y):\ x\in A,\ y\in B\}=\rho_l(A,B),$$

that is to say,

$$\rho_l(A,B)=f(\rho(A,B))\quad\text{for}\quad A,B\in E_0 \tag{23}$$

From (19) it follows that

$$(A\cap S_l(p,r)_{a(r)})\cup(B\cap S_l(p,r)_{b(r)})\subset C\cap S_l(p,r)_{\alpha(r)}.$$

Hence we get the inequality

$$\rho_l(A\cap S_l(p,r)_{a(r)},B\cap S_l(p,r)_{b(r)})$$

$$\leq d_l((A\cap S_l(p,r)_{a(r)})\cup(B\cap S_l(p,r)_{b(r)}))\leq d_l(C\cap S_l(p,r)_{\alpha(r)}) \tag{24}$$

From here, from (19), (24), from the properties of the function f and from the inequality

$$d_\rho(A\cup B)\leq d_\rho A+d_\rho B+\rho(A,B)\quad\text{for}\ \ A,B\in E_0 \tag{25}$$

we obtain

$$l(A\cap S_l(p,r)_{a(r)},B\cap S_l(p,r)_{b(r)})$$

$$\leq f(d_\rho((A\cap S_l(p,r)_{a(r)})\cup(B\cap S_l(p,r)_{b(r)})))$$

$$\leq f(d_\rho(A\cap S_l(p,r)_{a(r)})+d_\rho(B\cap S_l(p,r)_{b(r)})$$

$$+\rho(A\cap S_l(p,r)_{a(r)},B\cap S_l(p,r)_{b(r)}))$$

$$\leq f(d_\rho(A\cap S_l(p,r)_{a(r)}))+f(d_\rho(B\cap S_l(p,r)_{b(r)}))$$

$$+f(\rho(A\cap S_l(p,r)_{a(r)},B\cap S_l(p,r)_{b(r)}))$$

$$=d_l(A\cap S_l(p,r)_{a(r)})+d_l(B\cap S_l(p,r)_{b(r)})$$

$$+\rho_l(A\cap S_l(p,r)_{a(r)},B\cap S_l(p,r)_{b(r)})$$

$$\leq 3d_l(C\cap S_l(p,r)_{\alpha(r)})$$

in other words

$$l(A\cap S_l(p,r)_{a(r)},B\cap S_l(p,r)_{b(r)})\leq 3d_l(C\cap S_l(p,r)_{\alpha(r)}) \tag{26}$$

Hence and from the condition (18) we get

$$\frac{1}{r} l(A \cap S_l(p,r)_{a(r)}, B \cap S_l(p,r)_{b(r)}) \xrightarrow[r \to 0^+]{} 0 \tag{27}$$

Hence and from (27) it results that $(A,B) \in T_l(a,b,1,p)$. This ends the proof of the sufficient condition. The necessary condition of this theorem follows from the assumptions of the theorem and from Theorem 1.1 of this paper.

From Theorem 1.2, the following corollaries follow:

Corollary 1.2. *If* $l \in \mathfrak{F}_{f,\rho}$ *and* $A \cup B \in \widetilde{A}_p$*, then* $(A,B) \in T_l(a,b,1,p)$ *if and only if the functions* a, b *fulfil the condition* (11).

Corollary 1.3. *If* $l \in \mathfrak{F}_{f,\rho}$ *and* $A \in \widetilde{A}_p$ *is subsets of the arc* $B \in \widetilde{A}_p$*, then* $(A,B) \in T_l(a,b,1,p)$ *if and only if the functions* a, b *fulfil the condition* (11).

Corollary 1.4. *If* $A \in \widetilde{A}_p$ *and* $l \in \mathfrak{F}_{f,\rho}$*, then* $(A,A) \in T_l(a,b,1,p)$*, in other words, the tangency relation* $T_l(a,b,1,p)$ *is reflexive in the class* $\widetilde{A}_p$ *of the rectifiable arcs if and only if the functions* a, b *fulfil the condition* (11).

All results presented in this paper are true for the rectifiable arcs of the class A_p having the Archimedean property at the point p of the generalized metric space (E,l).

References

[1] Chądzyńska A., On some classes of sets related to the symmetry of the tangency relation in a metric space, Ann. Soc. Math. Polon., Comm. Math. 1972, 16, 219-228.

[2] Gołąb S., Moszner Z., Sur le contact des courbes dans les espaces metriques généraux, Colloq. Math. 1963, 10, 105-311.

[3] Grochulski J., Konik T., Tkacz M., On the tangency of sets in metric spaces, Ann. Polon. Math. 1980, 38, 121-131.

[4] Grochulski J., Some properties of tangency relations, Demonstratio Math. 1995, 28, 361-367.

[5] Konik T., On the tangency of sets of some class in generalized metric spaces, Demonstratio Math. 1989, 22(4), 1093-1107.

[6] Konik T., On the reflexivity symmetry and transitivity of the tangency relations of sets of the class $\widetilde{M}_{p,k}$, J. Geom. 1995, 52, 142-151

[7] Konik T., The compatibility of the tangency relations of sets in generalized metric spaces, Mat. Vesnik 1998, 50, 17-22.

[8] Konik T., On the compatibility and the equivalence of the tangency relations of sets of the classes $A^*_{p,k}$, J. Geom. 1998, 63, 124-133.

[9] Konik T., On some tangency relation of sets, Publ. Math. Debrecen 1999, 55/3-4, 411-419.

[10] Konik T., On the sets of the classes $\widetilde{M}_{p,k}$, Demonstratio Math. 2000, 33(2), 407-417.

[11] Waliszewski W., On the tangency of sets in a metric space, Colloq. Math. 1966, 15, 127-131

[12] Waliszewski W., O symetrii relacji styczności zbiorów w przestrzeni metrycznej, Zeszyty Nauk. Uniw. Łódz., Nauki Mat.-Przyr. 1966, seria II, 185-190.

[13] Waliszewski W., On the tangency of sets in generalized metric spaces, Ann. Polon. Math. 1973, 28, 275-284.