IDENTIFICATION OF INTERNAL HEAT SOURCE CAPACITY IN THE HETEROGENEOUS DOMAIN

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Abstract. The heterogeneous domain Ω being the composition of two domains (Ω_1, Ω_2) is considered. It is assumed that in the first domain Ω_1 the internal volumetric heat sources act. On the basis of knowledge of heating (cooling) curves at the selected set of Ω_2 the capacity of internal heat sources in Ω_1 is identified. The inverse problem formulated in this way is interesting from the practical point of view. For example, a such situation takes place during the casting solidification. The evolution of latent heat in the casting domain causes that in Ω_1 the internal heat sources appear, while in the mould domain this component of energy equation is equal to 0. Additionally the measurements of temperature in the mould are essentially simpler from the technical view-point. In the paper the theoretical base of the problem and the examples of numerical realization are discussed.

1. Mathematical description of the process

The temperature field in the domain Ω is described by the system of equations

$$c_e \frac{\partial T_e(x,t)}{\partial t} = \lambda_e \nabla^2 T_e(x,t) + Q_e, \quad e = 1, 2 \tag{1}$$

where $T_e(x, t)$ is the temperature in sub-domain Ω_e, c_e is the volumetric specific heat, λ_e is the thermal conductivity, x and t denote the spatial co-ordinates and time, correspondingly. Additionally it is assumed that $Q_2 = 0$.
On the contact surface between Ω_1 and Ω_2 the continuity condition is given

$$x \in \Gamma_{12}: -\lambda_1 \frac{\partial T_1(x,t)}{\partial n} = \frac{T_1(x,t) - T_2(x,t)}{R} = -\lambda_2 \frac{\partial T_2(x,t)}{\partial n} \tag{2}$$

where $\partial/\partial n$ denotes a normal derivative, R is the thermal resistance between Ω_1 and Ω_2. In the case of an ideal contact the last condition takes a form

$$x \in \Gamma_{12}: \begin{cases} -\lambda_1 \dfrac{\partial T_1(x,t)}{\partial n} = -\lambda_2 \dfrac{\partial T_2(x,t)}{\partial n} \\ T_1(x,t) = T_2(x,t) \end{cases} \tag{3}$$

On the outer surface of the system the Robin condition is taken into account

$$x \in \Gamma_0 : -\lambda_2 \frac{\partial T_2(x,t)}{\partial n} = \alpha\left[T_2(x,t) - T_a\right] = q_b \tag{4}$$

where α is the heat transfer coefficient, T_a is the ambient temperature.
For $t = 0$ the initial temperature field is known, namely

$$t = 0 : T_1(x,0) = T_{10}, T_2(x,0) = T_{20} \tag{5}$$

where T_{10} and T_{20} are the initial temperatures in sub-domains Ω_1 and Ω_2.
So, the following boundary-initial problem is considered

$$\begin{cases} x \in \Omega_1 : & c_1 \dfrac{\partial T_1}{\partial t} = \lambda_1 \nabla^2 T_1 + Q_1 \\ x \in \Omega_2 : & c_2 \dfrac{\partial T_2}{\partial t} = \lambda_2 \nabla^2 T_2 \\ x \in \Gamma_{12} : & \begin{cases} T_1 = T_2 \\ -\lambda_1 \dfrac{\partial T_1}{\partial n} = -\lambda_2 \dfrac{\partial T_2}{\partial n} \end{cases} \\ x \in \Gamma_0 : & -\lambda_2 \dfrac{\partial T_2}{\partial n} = \alpha (T_2 - T_a) \\ t = 0 : & T_1 = T_{10}, T_2 = T_{20} \end{cases} \tag{6}$$

The task above formulated allows to find the basic (direct) solution which is necessary in order to solve the inverse problem. The details concerning the identification of source function Q_1 will be discussed in the next chapter.

2. Identification of heat source

Let Z denotes the sensitivity function [1-3] $Z = \partial T/\partial Q$. The distribution of this function results from the solution of the following boundary-initial problem

$$\begin{cases} x \in \Omega_1 : & c_1 \dfrac{\partial Z_1}{\partial t} = \lambda_1 \nabla^2 Z_1 + 1 \\ x \in \Omega_2 : & c_2 \dfrac{\partial Z_2}{\partial t} = \lambda_2 \nabla^2 Z_2 \\ x \in \Gamma_{12} : & \begin{cases} Z_1 = Z_2 \\ -\lambda_1 \dfrac{\partial Z_1}{\partial n} = -\lambda_2 \dfrac{\partial Z_2}{\partial n} \end{cases} \\ x \in \Gamma_0 : & -\lambda_2 \dfrac{\partial Z_2}{\partial n} = \alpha Z_2 \\ t = 0 : & Z_1 = 0, Z_2 = 0 \end{cases} \tag{7}$$

A small difference in the model (7) appears if the thermal resistance $R \neq 0$.
Because the temperature field is continuous, it can be expanded in a Taylor series about an arbitrary but known value Q^*. For linear problems, only the first derivative is nonzero. Thus

$$T_i^f = \hat{T}_i^f + (Q - Q^*) \cdot \left. \frac{\partial T_i^f}{\partial Q} \right|_{Q=Q^*} \tag{8}$$

or

$$T_i^f = \hat{T}_i^f + Z_i^f \cdot (Q - Q^*) \tag{9}$$

The algorithm for estimating Q involves minimizing [1-3]

$$S = \sum_{f=1}^{F} \sum_{i=1}^{M} (T_i^f - T_{di}^f)^2 \tag{10}$$

Insertion (9) into (10) leads to the formula

$$S(Q) = \sum_{f=1}^{F} \sum_{i=1}^{M} (\hat{T}_i^f + Z_i^f \cdot (Q - Q^*) - T_{di}^f)^2 \tag{11}$$

The condition of the functional (11) minimum gives

$$\frac{dS(Q)}{dQ} = 2\sum_{f=1}^{F} \sum_{i=1}^{M} (\hat{T}_i^f + Z_i^f \cdot (Q - Q^*) - T_{di}^f) Z_i^f = 0 \tag{12}$$

After simple transformations we obtain

$$\hat{Q} = Q^* + \frac{\sum_{f=1}^{F} \sum_{i=1}^{M} Z_i^f (T_{di}^f - \hat{T}_i^f)}{\sum_{f=1}^{F} \sum_{i=1}^{M} (Z_i^f)^2} \tag{13}$$

where $\hat{Q}$ is an estimate of the unknown heat source.

3. Examples of computations

The presented solution concerns the 1D problem. The segment $x \in [0, 0.05]$ corresponds to sub-domain Ω_1, while the segment $x \in [0.05, 0.1]$ corresponds to Ω_2.

The thermophysical parameters of Ω_1 and Ω_2 equal [4] $c_1 = 4.5$ MJ/m^{3}K, $\lambda_1 = 25$ W/mK, $c_2 = 1.7$ MJ/m^{3}K, $\lambda_2 = 1.5$ W/mK. Thermal resistance $R = 0.5$ m^{2}K/W. For $x = 0$: $q_b = 0$, for $x = 0.1$ $\alpha = 150$ W/m^2 K, $T_a = 40^{\circ}$C. Initial temperatures of Ω_1 and Ω_2 equal 950°C. The value of identified source term $Q_1 = 10\ 000$ W/m^3 has been assumed. In Figure 1 the temperature profiles in domain Ω_2 for selected times are marked.

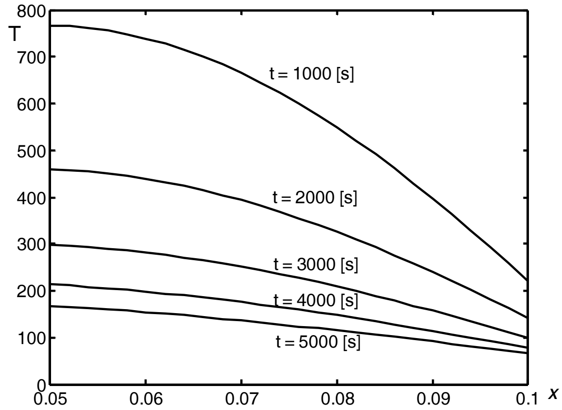

Fig. 1. Temperature profiles in Ω_2

The sensitivity functions in domain Ω_1 change with time in a linear manner. In Figure 2 the course of $U = Z \cdot 10^5$ for $x = 0.002$ is shown. The courses of function U at the set of selected points from Ω_2 are shown in Figure 3.

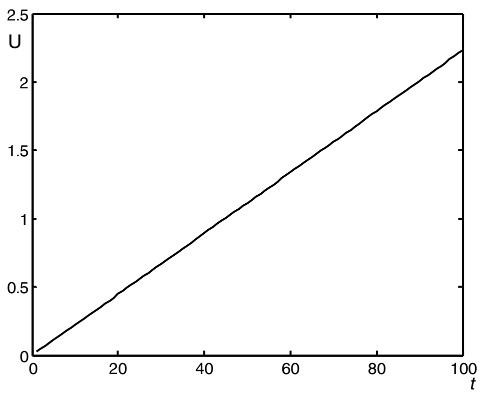

Fig. 2. The course of $U = Z \cdot 10^5$ for $x = 0.002$

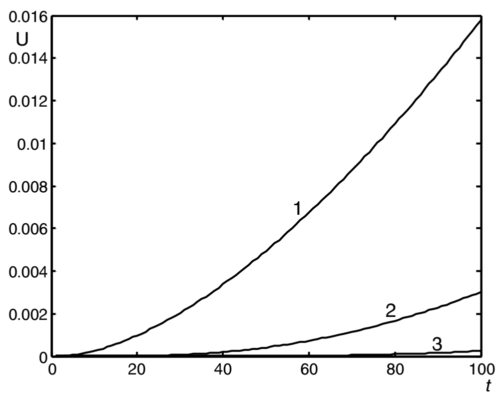

Fig. 3. The courses of $U = Z \cdot 10^5$ (1 - $x = 0.052$, 2 - $x = 0.062$, 3 - $x = 0.076$)

In numerical realization (FEM [5]) the domain Ω has been divided into 50 linear finite elements, time step $\Delta t = 1$ s. The cooling curves at the six control points (sensors, x_i = 0.052, 0.06, 0.068, 0.082, 0.09, 0.098) are registered until t = 5000 s. One can see that the control points have been located in Ω_2. The results presented have been obtained after 3 iterations (start point $Q_1^0 = 0$). In Table 1 the results of identification for undisturbed temperatures (1) and disturbed temperatures (2) (standard deviation $\sigma = 1$) are presented. The exact values of temperature have been found on the basis of direct problem solution for Q_1 = 10 000 W/m^3.

Table 1

Case	Number of simulation	$\hat{Q}_1$
(1) undisturbed	1	10000.0000000105
(2) disturbed	2	9902.39010983652
	3	10054.6491645594
	4	10032.2666852779
	5	9901.82198847453
	6	9879.76202753673
	7	10072.3062107584
	8	10080.1764011397
	9	9821.18162937548
	10	10070.1936981191
Mean value		9981.474792

In the second part of computations the sensors have been located close to the contact surface (x_i = 0.052, 0.054, 0.056, 0.058, 0.06, 0.062). The results of the same computations as previously are collected in Table 2.

Table 2

Case	Number of simulation	$\hat{Q}_1$
(1) undisturbed	1	10000.0000000094
(2) disturbed	2	10069.2175620262
	3	9969.97385610996
	4	9926.51279352947
	5	9954.12962328423
	6	9977.85661863327
	7	9936.62552787668
	8	10095.0554646099
	9	10065.1874372992
	10	9994.96145354483x
Mean value		9998.952034

In Table 3 one can see the identification of source term using only one sensor for $x = 0.052$. The estimation of Q_1 is sufficiently good.

Table 3

Case	Number of simulation	$\hat{Q}_1$
(1) undisturbed	1	9999.99999998933
(2) disturbed	2	9690.91715169449
	3	9772.6970197913
	4	10014.2785637644
	5	9873.5608442589
	6	9893.20484309091
	7	9895.01989919977
	8	9956.68484585822
	9	9883.59864863115
	10	9900.45371262416
Mean value		9875.601725

It should be pointed out that the authors of this paper have at one's disposal the solutions of similar inverse problems for the case of 2D domains. They will be presented in the doctoral theses prepared by A. Metelski.
Summing up, it is possible to identify the source term Q_1 in the heterogeneous domain on the basis of knowledge of temperature field only in the sub-domain for which $Q_2 = 0$. In the case of constant value of Q_1 the algorithm proposed is quite effective and exact, at the same time the number of iterations is rather small.

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