IDENTIFICATION OF BOUNDARY HEAT FLUX USING THE GLOBAL FUNCTION SPECIFICATION

Ewa Majchrzak [1, 2]***, Katarzyna Freus*** [2]***, Jerzy Mendakiewicz*** [1]

[1] *Silesian University of Technology, Gliwice*
[2] *Czestochowa University of Technology, Czestochowa*

Abstract. In the paper the inverse problem consisting in the identification of boundary heat flux is presented. On the basis of the knowledge of heating curves at the selected points from the domain the time dependent value of boundary heat flux is identified. In order to solve the problem the global function specification method has been used, on the stage of numerical computations the boundary element method has been applied. In the case of disturbed input data the regularization procedures have been introduced. The theoretical considerations are supplemented by the examples of computations.

1. Direct and inverse problem

The following boundary initial problem (1D thermal diffusion) is considered

$$\begin{cases} 0 < x < L: & \dfrac{\partial T(x,\ t)}{\partial t} = a \dfrac{\partial^2 T(x,\ t)}{\partial x^2} \\ x = 0: & q(x,\ t) = \lambda \dfrac{\partial T(x,\ t)}{\partial x} = 0 \\ x = L: & q(x,\ t) = -\lambda \dfrac{\partial T(x,\ t)}{\partial x} = q(t) \\ t = 0: & T(x,\ t) = T_0(x) \end{cases} \tag{1}$$

where $a = \lambda/c$ is the thermal diffusion coefficient (λ - thermal conductivity, c - volumetric specific heat), T is the temperature, x - geometrical co-ordinate, t - time. For $x = 0$ the no-flux condition is assumed, while for $x = L$ the time-dependent heat flux is known. For $t = 0$ the initial condition $T_0(x)$ is given.

In the direct problem described above, the parameters λ, c, L as well as the initial and boundary conditions are known. The objective of the direct problem is to determine the transient temperature field $T(x,t)$ in the body.

For the inverse problem considered here, for $x = L$ the time-dependent boundary heat flux $q(t)$ is regarded as unknown, while the other quantities appearing in the formulation of the direct problem (1) are assumed to be known.

In order to solve the inverse problem discussed the knowledge of additional information is necessary [1, 2]. In particular we assume that the cooling (heating) curves at the selected set of points (sensors) x_i are known:

$$T_{di}^{f} = T_d\left(x_i,\ t^f\right), \quad i = 1, 2, ..., M, \qquad f = 1, 2, ..., F \tag{2}$$

where M is a number of sensors. The temperature measurements (2) may contain the random errors. Such errors are assumed here to be normally distributed with a zero mean and a known constant standard deviation σ.

2. Global function specification method

Owning to the discrete nature of temperature data (2) the unknown function $q(t)$ must also be expressed in a discrete form [3, 4], for example (Fig. 1)

$$t \in \left[t^{f-1},\ t^f\right]: \quad q^f = q\left(t^f\right), \quad f = 1, 2, ..., F \tag{3}$$

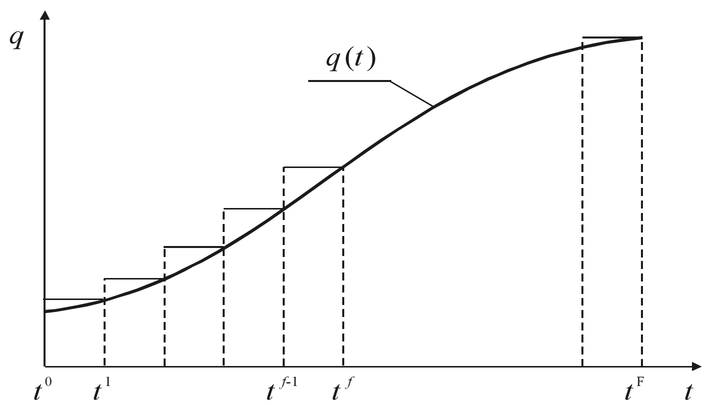

Fig. 1. Approximation of unknown boundary heat flux

In global function specification method the unknown values $q^1, q^2, ..., q^F$ are identified simultaneously [3, 4]. In this case the following least squares criterion is considered

$$S\left(q^1,\ q^2,\ ...,\ q^F\right) = \sum_{f=1}^{F} \sum_{i=1}^{M} \left(T_i^f - T_{di}^f\right)^2 \to \text{MIN} \tag{4}$$

where T_i^f, T_{di}^f are the estimated and measured temperatures, respectively, for time t^f, $f = 1, 2, ..., F$ and for the sensor x_i, $i = 1, 2, ..., M$.

At first, the direct problem (1) is solved under the assumption that $q^f = \hat{q}^f$, f = 1, 2,..., F, where $\hat{q}^f$ are the arbitrary assumed values of the heat flux. The solution obtained, this means the temperature distribution at the points x_i for times t^f, f = 1, 2,..., F we denote by $\hat{T}_i^f$.

Function T_i^f is expanded in a Taylor series

$$T_i^f = \hat{T}_i^f + \sum_{k=1}^{f} \left. \frac{\partial T_i^f}{\partial q^k} \right|_{q^k = \hat{q}^k} \left(q^k - \hat{q}^k \right) \tag{5}$$

Putting (5) into (4) one has

$$S\left(q^1,\ q^2,\ \ldots,\ q^F\right) = \sum_{f=1}^{F} \sum_{i=1}^{M} \left[\hat{T}_i^f + \sum_{k=1}^{f} \frac{\partial T_i^f}{\partial q^k} \left(q^k - \hat{q}^k \right) - T_{d\,i}^f \right]^2 \tag{6}$$

Using the necessary condition of function of several variables minimum one obtains

$$\frac{\partial S\left(q^1, q^2,\ \ldots,\ q^F\right)}{\partial q^p} = 2 \sum_{f=1}^{F} \sum_{i=1}^{M} \left[\hat{T}_i^f + \sum_{k=1}^{f} \frac{\partial T_i^f}{\partial q^k} \left(q^k - \hat{q}^k \right) - T_{d\,i}^f \right] \frac{\partial T_i^f}{\partial q^p} = 0 \tag{7}$$

where $p = 1,2,\ldots, F$.

Because for $f < p$: $\partial T_i^f / \partial q^p = 0$, so the system of equations (7) can be written in the form

$$\sum_{f=p}^{F} \sum_{i=1}^{M} \left[\hat{T}_i^f + \sum_{k=1}^{f} Z_i^{f,k} \left(q^k - \hat{q}^k \right) - T_{d\,i}^f \right] Z_i^{f,p} = 0\ , \quad p = 1,\ 2,\ \ldots, F \tag{8}$$

where

$$Z_i^{f,k} = \frac{\partial T_i^f}{\partial q^k}, \quad Z_i^{f,p} = \frac{\partial T_i^f}{\partial q^p} \tag{9}$$

are the sensitivity coefficients. After the mathematical manipulations one obtains

$$\sum_{f=p}^{F} \sum_{i=1}^{M} \sum_{k=1}^{f} Z_i^{f,k} Z_i^{f,p} \left(q^k - \hat{q}^k \right) = \sum_{f=p}^{F} \sum_{i=1}^{M} Z_i^{f,p} \left(T_{d\,i}^f - \hat{T}_i^f \right)^2, \quad p = 1,\ 2,\ \ldots,\ F \tag{10}$$

This system of equations allows to determine the values q^1, q^2,..., q^F.

We introduce the matrix of $M \cdot F$ rows and F columns

$$\mathbf{Z} = \begin{bmatrix} \mathbf{Z}_1 \\ \mathbf{Z}_2 \\ \ldots \\ \mathbf{Z}_M \end{bmatrix} \tag{11}$$

which contains M submatrices $F \times F$

$$\mathbf{Z}_i = \begin{bmatrix} Z_i^{1,1} & 0 & 0 & \ldots & 0 \\ Z_i^{2,1} & Z_i^{2,2} & 0 & \ldots & 0 \\ Z_i^{3,1} & Z_i^{3,2} & Z_i^{3,3} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ Z_i^{F,1} & Z_i^{F,2} & Z_i^{F,3} & \ldots & Z_i^{F,F} \end{bmatrix} \tag{12}$$

and then the system of equations (10) can be written as follows

$$\mathbf{Z}^{\mathrm{T}}\mathbf{Z}\ \mathbf{q} = \mathbf{Z}^{\mathrm{T}}\mathbf{Z}\ \hat{\mathbf{q}} + \mathbf{Z}^{\mathrm{T}}\left(\mathbf{T}_d - \hat{\mathbf{T}}\right) \tag{13}$$

where

$$\mathbf{T}_d = \begin{bmatrix} \mathbf{T}_{d\,1} \\ \mathbf{T}_{d\,2} \\ \ldots \\ \mathbf{T}_{d\,M} \end{bmatrix}, \quad \hat{\mathbf{T}} = \begin{bmatrix} \hat{\mathbf{T}}_1 \\ \hat{\mathbf{T}}_2 \\ \ldots \\ \hat{\mathbf{T}}_M \end{bmatrix} \tag{14}$$

while

$$\mathbf{T}_{d\,i} = \begin{bmatrix} T_{d\,i}^1 \\ T_{d\,i}^2 \\ \ldots \\ T_{d\,i}^F \end{bmatrix}, \quad \hat{\mathbf{T}}_i = \begin{bmatrix} \hat{T}_i^1 \\ \hat{T}_i^2 \\ \ldots \\ \hat{T}_i^F \end{bmatrix} \tag{15}$$

are the submatrices of $\mathbf{T}$ and $\mathbf{T}_d$ and

$$\mathbf{q} = \begin{bmatrix} q^1 \\ q^2 \\ \ldots \\ q^F \end{bmatrix}, \quad \hat{\mathbf{q}} = \begin{bmatrix} \hat{q}^1 \\ \hat{q}^2 \\ \ldots \\ \hat{q}^F \end{bmatrix} \tag{16}$$

In order to determine the sensitivity coefficients, at first the governing equations (1) are differentiated with respect to the unknown boundary heat flux q and then:

$$\begin{aligned}
&0 < x < L: && \frac{\partial Z(x,\ t)}{\partial t} = a \frac{\partial^2 Z(x,\ t)}{\partial x^2} \\
&x = 0: && W(x,\ t) = \lambda \frac{\partial Z(x,\ t)}{\partial x} = 0 \\
&x = L: && W(x,\ t) = -\lambda \frac{\partial Z(x,\ t)}{\partial x} = 1 \\
&t = 0: && Z(x,\ t) = 0
\end{aligned} \tag{17}$$

where

$$Z(x,\ t) = \frac{\partial T(x,\ t)}{\partial q} \tag{18}$$

Because the matrices (12) are equal

$$\mathbf{Z}_i = \begin{bmatrix} Z_i^{1,1} & 0 & 0 & \ldots & 0 \\ Z_i^{2,1} & Z_i^{1,1} & 0 & \ldots & 0 \\ Z_i^{3,1} & Z_i^{2,1} & Z_i^{1,1} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ Z_i^{F,1} & Z_i^{F-1,1} & Z_i^{F-2,1} & \ldots & Z_i^{1,1} \end{bmatrix} \tag{19}$$

so the elements of the these matrices can be calculated using the formulas

$$\begin{aligned}
&Z_i^{1,1} = Z\left(x_i,\ t^1\right) \\
&Z_i^{f,1} = Z\left(x_i,\ t^f\right) - Z\left(x_i,\ t^{f-1}\right), \quad f = 2,\ 3,\ \ldots,\ F
\end{aligned} \tag{20}$$

The basic problem (1) and additional one (17) have been solved using 1st scheme of the boundary element method [5, 6].

3. Regularization procedures

In order to avoid the fluctuations of inverse problem solution, the regularization procedure can be taken into account [3, 4, 7, 8]. In this case the least squares criterion (4) is supplemented by additional components, namely

$$S\left(q^1,\ q^2,\ \ldots,\ q^F\right)=\sum_{f=1}^{F}\sum_{i=1}^{M}\left(T_i^f-T_{di}^f\right)^2+$$

$$\gamma\left[\alpha_0\sum_{f=1}^{F}\left(q^f\right)^2+\alpha_1\sum_{f=1}^{F-1}\left(\frac{\left(q^{f+1}-q^f\right)}{\Delta t}\right)^2+\alpha_2\sum_{f=1}^{F-2}\left(\frac{\left(q^{f+2}-2q^{f+1}+q^f\right)}{\Delta t^2}\right)^2\right]\rightarrow \mathrm{MIN} \tag{21}$$

where γ is the regularization parameter. The coefficients α_0, α_1, α_2 are connected with adequate order of regularization. If $\alpha_0 = 1$, $\alpha_1 = 0$ and $\alpha_2 = 0$ then zeroth order regularization is considered, if $\alpha_0 = 0$ or $\alpha_0 = 1$ and $\alpha_1 = 1$, $\alpha_2 = 0$ then first order regularization is introduced, while for the second order of regularization: $\alpha_0 = 0$ or $\alpha_0 = 1$, $\alpha_1 = 0$ or $\alpha_1 = 1$ and $\alpha_2 = 1$.
In the case of zeroth order regularization one obtains the following system of equations (c.f. equation (13))

$$\left(\mathbf{Z}^{\mathrm{T}}\mathbf{Z}+\gamma\,\mathbf{I}\right)\mathbf{q}=\mathbf{Z}^{\mathrm{T}}\mathbf{Z}\,\hat{\mathbf{q}}+\mathbf{Z}^{\mathrm{T}}\left(\mathbf{T}_d-\hat{\mathbf{T}}\right) \tag{22}$$

where **I** is the identity matrix.
For first order regularization one has

$$\left(\mathbf{Z}^{\mathrm{T}}\mathbf{Z}+\frac{\gamma}{\Delta t^2}\mathbf{W}_1\right)\mathbf{q}=\mathbf{Z}^{\mathrm{T}}\mathbf{Z}\,\hat{\mathbf{q}}+\mathbf{Z}^{\mathrm{T}}\left(\mathbf{T}_d-\hat{\mathbf{T}}\right) \tag{23}$$

where

$$\mathbf{W}_1=\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & 0 & \ldots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & -1 & 1 \end{bmatrix} \tag{24}$$

For second order regularization the system of equations takes a form

$$\left(\mathbf{Z}^{\mathrm{T}}\mathbf{Z}+\frac{\gamma}{\Delta t^4}\mathbf{W}_2\right)\mathbf{q}=\mathbf{Z}^{\mathrm{T}}\mathbf{Z}\,\hat{\mathbf{q}}+\mathbf{Z}^{\mathrm{T}}\left(\mathbf{T}_d-\hat{\mathbf{T}}\right) \tag{25}$$

where

$$\mathbf{W}_2 = \begin{bmatrix} 1 & -2 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ -2 & 5 & -4 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 & \dots & 0 & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 & -4 & 6 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & -4 & 5 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & -2 & 1 \end{bmatrix} \tag{26}$$

4. Results of computations

The basis of algorithms verification of boundary heat flux identification is the solution of direct problem (1) under the assumption that $q(\text{t}) = 2000 + 240t - 6t^2$. It is assumed that $L = 0.02$ m, $\lambda = 1$ W/mK, $c = 10^6$ J/m^{3}K and $T_0 = 100^\circ$C. The domain has been divided into 20 linear internal cells, time step: $\Delta t = 1$s.

In Figure 2 the cooling curves at the points $x_1 = 0.017$ m, $x_2 = 0.018$ m and $x_3 = 0.019$ m both in the case of exact solution as well as disturbed in the random way solution ($\sigma = 0.5$) are shown. In Figures 3 and 4 the results of inverse problem solution obtained using the global function specification method are shown.

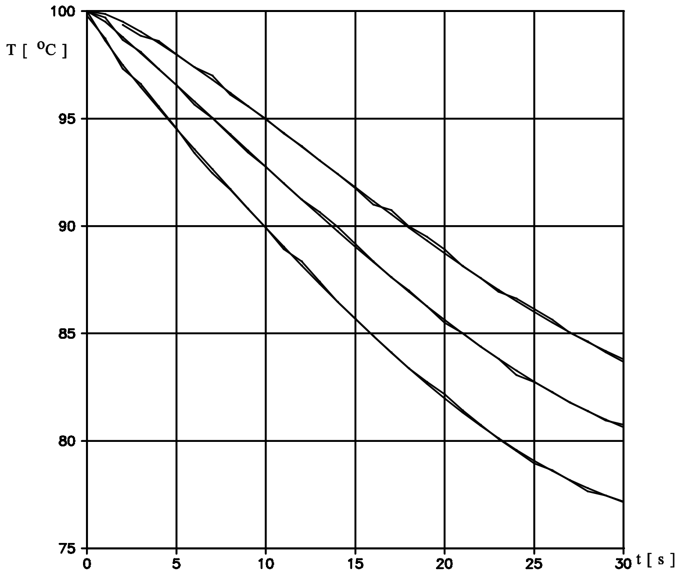

Fig. 2. Cooling curves (exact and disturbed) at the point x_1, x_2, x_3

For undisturbed cooling curves the exact values q^1, q^2,..., q^{30} have been obtained. For disturbed data the big oscillations of solution have been observed (Fig. 3).

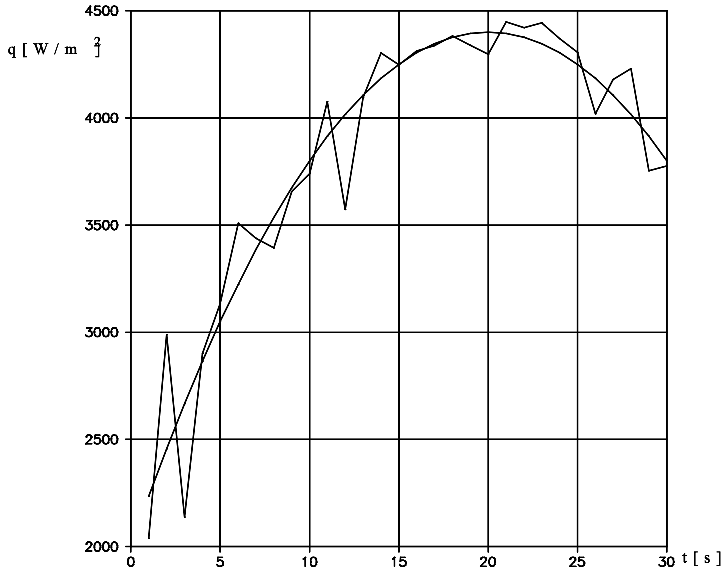

Fig. 3. Real and identified heat flux (without the regularization)

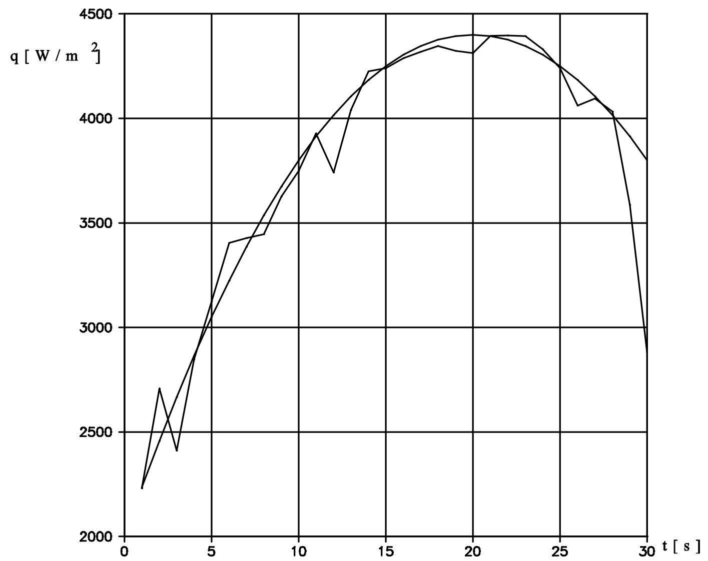

Fig. 4. Real and identified heat flux - regularization of zeroth order ($\gamma = 2 \cdot 10^{-7}$)

In Figure 4 the results obtained for zeroth order regularization ($\gamma = 2 \cdot 10^{-7}$) have been shown. In the case of regularization procedure application, the best results have been achieved for second order regularization ($\alpha_0 = 0$, $\alpha_1 = \alpha_2 = 1$) - Figure 5.

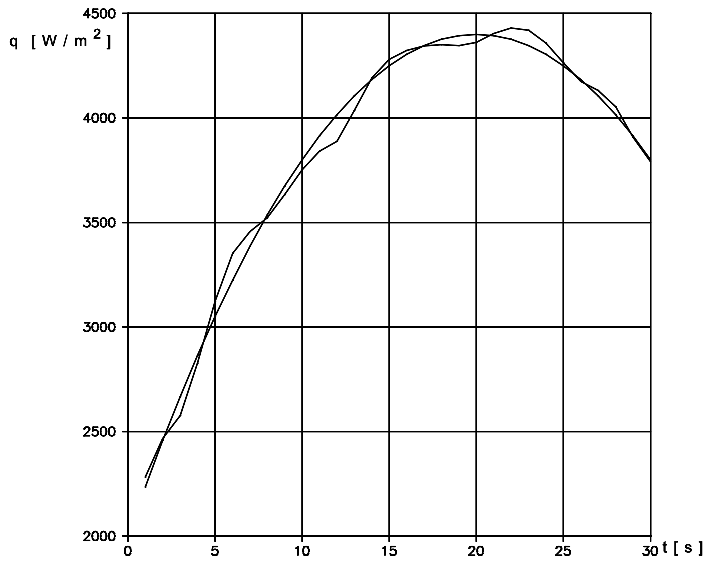

Fig. 5. Real and identified heat flux - regularization of second order ($\gamma = 2 \cdot 10^{-7}$, $\alpha_0 = 0$, $\alpha_1 = \alpha_2 = 1$)

Summing up, for undisturbed data the global function specification method leads to the exact identification of boundary heat flux. In the case of disturbed data the regilarization procedure should be introduced, the proper choice of regularization coefficient γ and coefficients α_0, α_1, α_2 is not simple [3, 9].

Acknowledgement

This paper has been sponsored by KBN (Grant No 3 T08B 004 28)

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