APPLICATION OF BOUNDARY ELEMENT METHOD TO SHAPE SENSITIVITY ANALYSIS

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Abstract. In the paper the boundary element method to shape sensitivity analysis is applied. The Laplace equation is analyzed and the aim of investigations is to estimate the changes of temperature in the 2D domain due to the change of local geometry of the boundary. Here the implicit differentiation method of shape sensitivity analysis is used. In the final part of the paper the example of numerical computations is shown.

1. Boundary element method for the Laplace equation

We consider the Laplace equation

$$(x,y)\in\Omega:\ \lambda\frac{\partial^2 T(x,y)}{\partial x^2}+\lambda\frac{\partial^2 T(x,y)}{\partial y^2}=0 \tag{1}$$

where λ [W/mK] is the thermal conductivity, T is the temperature, x, y are the geometrical co-ordinates. The equation (1) is supplemented by boundary conditions:

$$\begin{aligned}(x,y)\in\Gamma_1:&\quad T(x,y)=T_b\\ (x,y)\in\Gamma_2:&\quad q(x,y)=-\lambda\mathbf{n}\cdot\nabla T(x,y)=q_b\end{aligned} \tag{2}$$

where T_b is the known boundary temperature, q_b is the known boundary heat flux. In the case considered the boundary integral equation is following [l, 2]

$$\begin{aligned}(\xi,\eta)\in\Gamma:\ B(\xi,\eta)T(\xi,\eta)+\int_\Gamma T^*(x,\eta,x,y)q(x,y)\,\mathrm{d}\Gamma=\\ \int_\Gamma q^*(x,\eta,x,y)T(x,y)\,\mathrm{d}\Gamma\end{aligned} \tag{3}$$

where $B(\xi,\eta)\in(0,1)$ is the coefficient connected with the local shape of boundary, (ξ,η) is the observation point, $q(x,y)=-\lambda\mathbf{n}\cdot\nabla T(x,y)$, $T^*(\xi,\eta,x,y)$ is the fundamental solution

$$T^*(\xi,\eta,x,y)=\frac{1}{2\pi\lambda}\ln\frac{1}{r} \tag{4}$$

where r is the distance between the points (ξ,η) and (x, y)

$$r=\sqrt{(x-\xi)^2+(y-\eta)^2} \tag{5}$$

Function $q^*(\xi,\eta,x,y)$ is defined as follows

$$q^*(\xi,\eta,x,y)=-\lambda\mathbf{n}\cdot\nabla T^*(\xi,\eta,x,y) \tag{6}$$

and it can be calculated analytically

$$q^*(\xi,\eta,x,y)=\frac{d}{2\pi r^2} \tag{7}$$

where

$$d=(x-\xi)n_x+(y-\eta)n_y \tag{8}$$

while n_x, n_y are the directional cosines of the normal outward vector **n**.
In numerical realization of the BEM the boundary is divided into N boundary elements and integrals appearing in equation (3) are substituted by the sums of integrals over these elements

$$\begin{gathered}B(\xi_i,\eta_i)T(\xi_i,\eta_i)+\sum_{j=1}^{N}\int_{\Gamma_j}q(x,y)T^*(\xi_i,\eta_i,x,y)\,\mathrm{d}\Gamma_j=\\ \sum_{j=1}^{N}\int_{\Gamma_j}T(x,y)q^*(\xi_i,\eta_i,x,y)\,\mathrm{d}\Gamma_j\end{gathered} \tag{9}$$

Each point on the linear boundary element Γ_j is expressed as follows (Fig. 1)

$$(x,y)\in\Gamma_j:\begin{cases}x=N_p x_p^j+N_k x_k^j\\ y=N_p y_p^j+N_k y_k^j\end{cases} \tag{10}$$

where (x_p^j,y_p^j) and (x_k^j,y_k^j) correspond to the beginning and end of Γ_j while

$$N_p=\frac{1-\theta}{2},\quad N_k=\frac{1+\theta}{2} \tag{11}$$

are the shape functions $(\theta\in[-1,1])$.

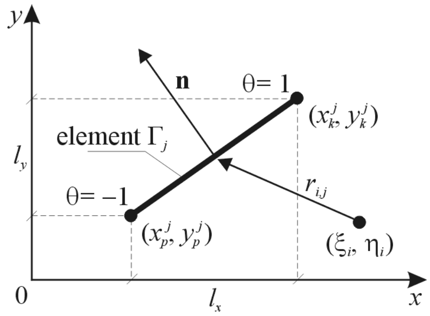

Fig. 1. Linear boundary element

It is easy to check that

$$\mathrm{d}\Gamma_j = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta = \sqrt{\left(\frac{x_k^j - x_p^j}{2}\right)^2 + \left(\frac{y_k^j - y_p^j}{2}\right)^2}\,\mathrm{d}\theta = \frac{l_j}{2}\mathrm{d}\theta \tag{12}$$

and

$$n_x^j = \frac{y_k^j - y_p^j}{l_j} = \frac{l_y^j}{l_j}, \qquad n_y^j = \frac{x_p^j - x_k^j}{l_j} = -\frac{l_x^j}{l_j} \tag{13}$$

where l_j is the length of the element Γ_j.
For the linear boundary element Γ_j, we assume that

$$(x, y) \in \Gamma_j : \begin{cases} T(\theta) = N_p T_p^j + N_k T_k^j \\ q(\theta) = N_p q_p^j + N_k q_k^j \end{cases} \tag{14}$$

The integrals appearing in equation (9) can be written in the form

$$\int_{\Gamma_j} q^*(\xi_i, \eta_i, x, y) T(x, y)\,\mathrm{d}\Gamma_j = \hat{H}_{ij}^p T_p^j + \hat{H}_{ij}^k T_k^j \tag{15}$$

and

$$\int_{\Gamma_j} T^*(\xi_i, \eta_i, x, y) q(x, y)\,\mathrm{d}\Gamma_j = G_{ij}^p q_p^j + G_{ij}^k q_k^j \tag{16}$$

where (c.f. equations (7), (8), (13)):

$$\hat{H}_{ij}^p = \frac{1}{4\pi}\int_{-1}^{1} N_p \frac{r_x^j l_y^j - r_y^j l_x^j}{r_{ij}^2}\,\mathrm{d}\theta \tag{17}$$

$$\hat{H}_{ij}^{k} = \frac{1}{4\pi}\int_{-1}^{1} N_k \frac{r_x^j l_y^j - r_y^j l_x^j}{r_{ij}^2}\, \mathrm{d}\theta \tag{18}$$

while (c.f. equation (4)):

$$G_{ij}^{p} = \frac{l_j}{4\pi\lambda}\int_{-1}^{1} N_p \ln\frac{1}{r_{ij}}\, \mathrm{d}\theta \tag{19}$$

$$G_{ij}^{k} = \frac{l_j}{4\pi\lambda}\int_{-1}^{1} N_k \ln\frac{1}{r_{ij}}\, \mathrm{d}\theta \tag{20}$$

In equations (17)-(20)

$$l_j = \sqrt{\left(x_k^j - x_p^j\right)^2 + \left(y_k^j - y_p^j\right)^2} = \sqrt{\left(l_x^j\right)^2 + \left(l_y^j\right)^2} \tag{21}$$

and

$$r_{ij} = \sqrt{\left(N_p x_p^j + N_k x_k^j - \xi_i\right)^2 + \left(N_p y_p^j + N_k y_k^j - \eta_i\right)^2} = \sqrt{\left(r_x^j\right)^2 + \left(r_y^j\right)^2} \tag{22}$$

As is well known, in the final system of algebraic equations the values of temperatures or heat fluxes are connected with the boundary nodes. If the following numeration of the boundary nodes $r = 1, 2,..., R$ is accepted then for $i = 1, 2,.... R$ one obtains the system of equations (c.f. equation (9))

$$B_i T_i + \sum_{r=1}^{R} G_{ir} q_r = \sum_{r=1}^{R} \hat{H}_{ir} T_r \tag{23}$$

where for the single node r being the end of the boundary element Γ_j and being the beginning of the boundary element Γ_{j+1} (Fig. 2) we have

$$\begin{aligned} G_{ir} &= G_{ij}^{k} + G_{i\,j+1}^{p} \\ \hat{H}_{ir} &= \hat{H}_{ij}^{k} + \hat{H}_{i\,j+1}^{p} \end{aligned} \tag{24}$$

while for double node r, $r+1$:

$$\begin{aligned} G_{ir} &= G_{ij}^{k}, \quad G_{i\,r+1} = G_{i\,j+1}^{p} \\ \hat{H}_{ir} &= \hat{H}_{ij}^{k}, \quad \hat{H}_{i\,r+1} = \hat{H}_{i\,j+1}^{p} \end{aligned} \tag{25}$$

The system of equations (23) can be written in the form

$$\sum_{r=1}^{R} G_{ir} q_r = \sum_{r=1}^{R} H_{ir} T_r, \quad i = 1,2,...,R \tag{26}$$

or

$$\mathbf{G}\mathbf{q} = \mathbf{H}\mathbf{T} \tag{27}$$

where

$$H_{ir} = \begin{cases} \hat{H}_{ir}, & i \neq r \\ \hat{H}_{ii} - B_i & i = r \end{cases} \tag{28}$$

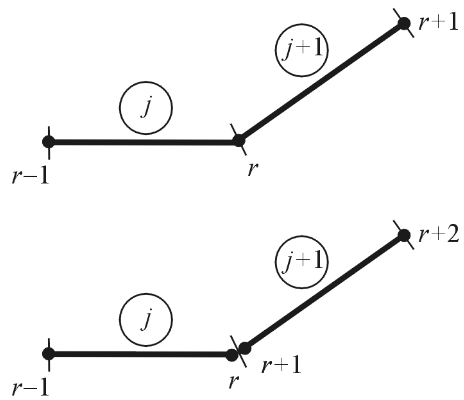

Fig. 2. Single and double nodes

It should be pointed out that it is convenient to calculate the values H_{ii} using the formula

$$H_{ii} = -\sum_{\substack{r=1 \\ r \neq i}}^{R} H_{ir}, \quad i = 1, 2, ..., R \tag{29}$$

Taking into account the boundary conditions (2) the system of equations (27) should be rebuilt to the form $\mathbf{A}\,\mathbf{Y} = \mathbf{F}$. The solution of this system allows to determine the "missing" boundary temperatures and heat fluxes. Next, the temperatures at optional set of internal nodes can be calculated using the formula

$$T_i = \sum_{r=1}^{R} H_{ir} T_r - \sum_{r=1}^{R} G_{ir} q_r + \sum_{r=1}^{R} Z_{ir} \tag{30}$$

2. Implicit differentiation method

We assume that b is the shape parameter, this means b corresponds to the x or y coordinate of one of boundary node. The implicit differentiation method [3] starts with the algebraic system of equations (27). The differentiation of (27) with respect to b leads to the following system of equations

$$\frac{\mathrm{D}\mathbf{G}}{\mathrm{D}b}\mathbf{q}+\mathbf{G}\frac{\mathrm{D}\mathbf{q}}{\mathrm{D}b}=\frac{\mathrm{D}\mathbf{H}}{\mathrm{D}b}\mathbf{T}+\mathbf{H}\frac{\mathrm{D}\mathbf{T}}{\mathrm{D}b} \tag{31}$$

from which

$$\mathbf{G}\frac{\mathrm{D}\mathbf{q}}{\mathrm{D}b}=\mathbf{H}\frac{\mathrm{D}\mathbf{T}}{\mathrm{D}b}+\frac{\mathrm{D}\mathbf{H}}{\mathrm{D}b}\mathbf{T}-\frac{\mathrm{D}\mathbf{G}}{\mathrm{D}b}\mathbf{q} \tag{32}$$

Differentiation of boundary conditions (2) gives:

$$\begin{aligned} &(x,y)\in\Gamma_1: \quad \frac{\mathrm{D}T}{\mathrm{D}b}=0\\ &(x,y)\in\Gamma_2: \quad \frac{\mathrm{D}q}{\mathrm{D}b}=0 \end{aligned} \tag{33}$$

So, this approach of shape sensitivity analysis is connected with the differentiation of elements of matrices **G** and **H**.
Let $b = x_r$ where (x_r, y_r) is the single boundary node (c.f. Fig. 2). Because

$$x_r = x_k^j = x_p^{j+1} \tag{34}$$

so after differentiation of **G** we obtain the following non-zero elements

$$\frac{\partial G_{ij}^p}{\partial x_k^j},\ \frac{\partial G_{ij}^k}{\partial x_k^j},\ \frac{\partial G_{i\,j+1}^p}{\partial x_p^{j+1}},\ \frac{\partial G_{i\,j+1}^k}{\partial x_p^{j+1}} \tag{35}$$

and

$$\frac{\partial G_{rs}^p}{\partial \xi_r},\ \frac{\partial G_{rs}^k}{\partial \xi_r} \tag{36}$$

where $s = 1, 2,..., j-1, j+2,..., N$.
Taking into account the formulas (19), (20) one has:

$$\frac{\partial G_{ij}^p}{\partial x_k^j}=\frac{1}{4\pi\lambda}\frac{l_x^j}{l_j}\int_{-1}^{1}N_p\ln\frac{1}{r_{ij}}\mathrm{d}\theta-\frac{l_j}{4\pi\lambda}\int_{-1}^{1}N_p\frac{\left(N_k-\delta_{ir}\right)r_x^j}{r_{ij}^2}\mathrm{d}\theta \tag{37}$$

$$\frac{\partial G_{ij}^k}{\partial x_k^j}=\frac{1}{4\pi\lambda}\frac{l_x^j}{l_j}\int_{-1}^{1}N_k\ln\frac{1}{r_{ij}}\mathrm{d}\theta-\frac{l_j}{4\pi\lambda}\int_{-1}^{1}N_k\frac{\left(N_k-\delta_{ir}\right)r_x^j}{r_{ij}^2}\mathrm{d}\theta \tag{38}$$

$$\frac{\partial G_{i\,j+1}^{p}}{\partial x_k^{j+1}} = -\frac{1}{4\pi\lambda}\frac{l_x^{j+1}}{l_{j+1}}\int_{-1}^{1} N_p \ln\frac{1}{r_{i\,j+1}}\mathrm{d}\theta - \frac{l_{j+1}}{4\pi\lambda}\int_{-1}^{1} N_p \frac{\left(N_p-\delta_{ir}\right)r_x^{j+1}}{r_{i\,j+1}^2}\mathrm{d}\theta \tag{39}$$

$$\frac{\partial G_{i\,j+1}^{k}}{\partial x_p^{j+1}} = -\frac{1}{4\pi\lambda}\frac{l_x^{j+1}}{l_{j+1}}\int_{-1}^{1} N_k \ln\frac{1}{r_{i\,j+1}}\mathrm{d}\theta - \frac{l_{j+1}}{4\pi\lambda}\int_{-1}^{1} N_k \frac{\left(N_p-\delta_{ir}\right)r_x^{j+1}}{r_{i\,j+1}^2}\mathrm{d}\theta \tag{40}$$

and for $s \neq j$ and $s \neq j+1$:

$$\frac{\partial G_{rs}^{p}}{\partial \xi_r} = \frac{l_s}{4\pi\lambda}\int_{-1}^{1} N_p \frac{r_x^s}{r_{rs}^2}\mathrm{d}\theta \tag{41}$$

$$\frac{\partial G_{rs}^{k}}{\partial \xi_r} = \frac{l_s}{4\pi\lambda}\int_{-1}^{1} N_k \frac{r_x^s}{r_{rs}^2}\mathrm{d}\theta \tag{42}$$

In similar way, the differentiation of **H** leads to the following non-zero elements:

$$\frac{\partial \hat{H}_{ij}^{p}}{\partial x_k^{j}} = \frac{1}{4\pi}\int_{-1}^{1} N_p \frac{\left[\left(N_k-\delta_{ir}\right)l_y^j - r_y^j\right]r_{ij}^2 + 2r_x^j\left(\delta_{ir}-N_k\right)c^j}{r_{ij}^4}\mathrm{d}\theta \tag{43}$$

$$\frac{\partial \hat{H}_{ij}^{k}}{\partial x_k^{j}} = \frac{1}{4\pi}\int_{-1}^{1} N_k \frac{\left[\left(N_k-\delta_{ir}\right)l_y^j - r_y^j\right]r_{ij}^2 + 2r_x^j\left(\delta_{ir}-N_k\right)c^j}{r_{ij}^4}\mathrm{d}\theta \tag{44}$$

$$\frac{\partial \hat{H}_{i\,j+1}^{p}}{\partial x_p^{j+1}} = \frac{1}{4\pi}\int_{-1}^{1} N_p \frac{\left[\left(N_p-\delta_{ir}\right)l_y^{j+1} - r_y^{j+1}\right]r_{i\,j+1}^2 + 2r_x^{j+1}\left(\delta_{ir}-N_p\right)c^{j+1}}{r_{i\,j+1}^4}\mathrm{d}\theta \tag{45}$$

$$\frac{\partial \hat{H}_{i\,j+1}^{k}}{\partial x_p^{j+1}} = \frac{1}{4\pi}\int_{-1}^{1} N_k \frac{\left[\left(N_p-\delta_{ir}\right)l_y^{j+1} - r_y^{j+1}\right]r_{i\,j+1}^2 + 2r_x^{j+1}\left(\delta_{ir}-N_p\right)c^{j+1}}{r_{i\,j+1}^4}\mathrm{d}\theta \tag{46}$$

and for $s \neq j$ and $s \neq j+1$:

$$\frac{\partial \hat{H}_{rs}^{p}}{\partial \xi_r} = \frac{1}{4\pi}\int_{-1}^{1} N_p \frac{-l_y^s r_{rs}^2 + 2r_x^s c^s}{r_{rs}^4}\mathrm{d}\theta \tag{47}$$

$$\frac{\partial \hat{H}_{rs}^{k}}{\partial \xi_r} = \frac{1}{4\pi}\int_{-1}^{1} N_k \frac{-l_y^s r_{rs}^2 + 2r_x^s c^s}{r_{rs}^4}\mathrm{d}\theta \tag{48}$$

where

$$c^j = r_x^j l_y^j - r_y^j l_x^j \tag{49}$$

The way of the matrices $\partial G_{ir} / \partial x_r$ and $\partial H_{ir} / \partial x_r$ creation is similar as for the matrices G_{ir} and H_{ir} (c.f. equations (24)) and the non-zero elements of these matrices appear in the columns r–1, r, r+1 and in the row r.

3. Example of computations

The square of dimensions 0.05×0.05 m has been considered. Thermal conductivity equals $\lambda = 1$ W/(mK). On the bottom boundary the Neumann condition $q_b = -10^4$ W/m^2 has been assumed, on the remaining parts of the boundary the Dirichlet condition $T_b = 500$°C has been accepted. The boundary has been divided into 8 linear boundary elements (Fig. 3) and 10 boundary nodes have been distinguished (two double boundary nodes).

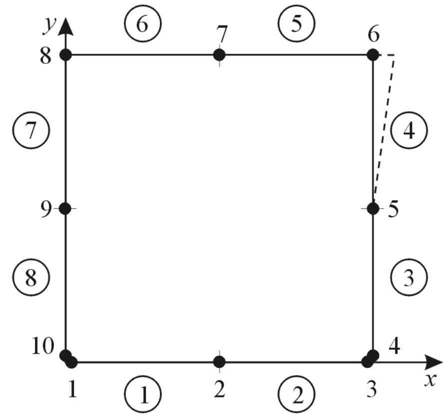

Fig. 3. Discretization

The solution of basic problem (equations (1), (2)) is following

$$\mathbf{T} = \begin{bmatrix} 500 \\ 676.40 \\ 500 \\ 500 \\ 500 \\ 500 \\ 500 \\ 500 \\ 500 \\ 500 \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} -10000 \\ -10000 \\ -10000 \\ 19122.08 \\ -324.51 \\ 251.10 \\ 938.72 \\ 251.10 \\ -324.51 \\ 19122.08 \end{bmatrix} \tag{50}$$

For the shape parameter $b = x_6$ we have

$$\frac{\partial \mathbf{G}^p}{\partial x_6} = \begin{bmatrix} 0 & 0 & 0 & -0.009 & 0.204 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.008 & 0.225 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.240 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.240 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.299 & 0 & 0 & 0 \\ -0.019 & -0.012 & 0 & 0 & 0.331 & -0.061 & -0.038 & -0.028 \\ 0 & 0 & 0 & -0.021 & 0.254 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.012 & 0.223 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.012 & 0.218 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.009 & 0.204 & 0 & 0 & 0 \end{bmatrix} \tag{51}$$

$$\frac{\partial \mathbf{G}^k}{\partial x_6} = \begin{bmatrix} 0 & 0 & 0 & -0.015 & 0.217 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.013 & 0.234 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.237 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.237 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.288 & 0 & 0 & 0 \\ -0.018 & -0.006 & 0 & 0 & 0.254 & -0.049 & -0.036 & -0.024 \\ 0 & 0 & 0 & -0.049 & 0.331 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.026 & 0.254 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.023 & 0.241 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.015 & 0.217 & 0 & 0 & 0 \end{bmatrix} \tag{52}$$

$$\frac{\partial \mathbf{H}^p}{\partial x_6} = \begin{bmatrix} 0 & 0 & 0 & -0.773 & 0.648 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.401 & 1.126 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.953 & 1.652 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.953 & 1.652 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3.573 & 0 & 0 & 0 \\ -0.457 & -0.405 & -1.230 & 0 & 0 & 0 & -0.710 & -0.223 \\ 0 & 0 & 0 & 0.840 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.247 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.463 & 0.405 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.773 & 0.648 & 0 & 0 & 0 \end{bmatrix} \tag{53}$$

and

$$\frac{\partial \mathbf{H}^k}{\partial x_6}=\begin{bmatrix} 0 & 0 & 0 & -0.818 & 0.944 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.145 & 1.421 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.230 & 1.531 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.230 & 1.531 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2.794 & 0 & 0 & 0 \\ -0.498 & -0.231 & -1.953 & 0 & 0 & 0 & -0.563 & -0.095 \\ 0 & 0 & 0 & -0.840 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.247 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.810 & 0.868 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.818 & 0.944 & 0 & 0 & 0 \end{bmatrix} \tag{54}$$

Finally, one obtains the following solution of additional problem (32), (33)

$$\frac{\mathrm{D}\mathbf{T}}{\mathrm{D}b}=\begin{bmatrix} 0 \\ 0.43346 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \frac{\mathrm{D}\mathbf{q}}{\mathrm{D}b}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ -1490.11 \\ 5676.25 \\ -30567.38 \\ 2117.76 \\ -384.28 \\ 140.50 \\ 15.85 \end{bmatrix} \tag{55}$$

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