APPLICATION OF GREEN'S FUNCTION METHOD IN FREQUENCY ANALYSIS OF AXISYMMETRIC VIBRATION OF ANNULAR PLATES WITH ELASTIC RING SUPPORTS

Stanisław Kukla, Mariusz Szewczyk

Institute of Mathematics and Computer Science, Czestochowa University of Technology

Abstract. The paper concerns the axisymmetric vibration of annular plates with concentric elastic ring supports. The formulation and solution to the problem take into account the plates with one edge clamped and another one free and arbitrary number of circular supports. The Green's function method is applied to obtain the exact solution of the problem in an analytical form. Presented numerical examples show the influence of the parameters characterizing the system on its free vibration frequencies.

Introduction

The axisymmetric vibrations of circular and annular plates have been widely discussed in literature (see, e.q. references [1-7]). The free vibration of circular plates with elastic ring supports was considered in papers [1-3, 6] and the vibration of annular plates with such supports was investigated in references [2, 4, 5, 7]. In papers [1, 4] the solutions to the problems have been obtained by using the Rayleigh-Ritz method and in [1, 2] the finite element solution is presented. The exact solutions to the eigenproblems are given in the papers [3, 5].

Solution to the vibration problem of circular plates includes the Bessel functions of the first and second kind, but the solution to the problem of annular plates is expressed by the Bessel functions and modified Bessel functions of first and second kinds. For this reason the approaches to solutions are separately determined for the two kinds of the plates. Numerical results yet, for instance, the eigenfrequencies, of the annular plate with free inner edge of small inner radius are approximately equal the eigenfrequencies of the circular plate with the same outer radius, thickness and boundary conditions at the outer edge as the annular plate [5].

The solution to free vibration problem of circular and annular plates with elastic ring supports can be obtained by using Green's function method. The method has been applied in frequency analysis of a circular plate in paper [6] and annular plate in paper [7]. The investigations presented in the paper [7] concern the annular plate with free inner and outer edges. In this paper the transverse vibration of a plate with inner edge clamped and outer free (C-F) is considered. The solution for the plate with inner edge free and outer clamped is obtained by changing the variables and parameters in solution for C-F plate. The Green's functions corresponding to the two cases of the plates are determined. Numerical examples show the effect of the elastic concentric ring on the eigenfrequencies of the system.

1. Formulation and solution to the problem

Let's consider an annular plate with uniform thickness. The problem of axisymmetric vibration of this plate is governed by a differential equation

$$D\frac{\partial}{\partial r}\left\{r\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)\right]\right\}=qr-\bar{\rho}hr\frac{\partial^2 w}{\partial t^2} \tag{1}$$

where w is the transverse displacement of a plate, r is the radial variable, t is the time variable, D is the bending rigidity of the plate, $\bar{\rho}$ is the mass per unit volume, q is the load per unit area and h is the plate thickness. For the plate supported by elastic concentric rings with radii r_j (j = 1,…, n) function q is

$$q(r,t)=-\sum_{j=1}^{n}k_j w(r,t)\,\delta(r-r_j)$$

where k_j's are stiffness coefficients of the elastic rings and δ is the Dirac delta function. For free vibrations one assumes: $w(r,t)=W(r)\,e^{i\omega t}$ and equation (1) takes the form

$$\frac{\mathrm{d}}{\mathrm{d}r}\left\{r\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}W}{\mathrm{d}r}\right)\right]\right\}-r\,\Omega^4 W=-\sum_{j=1}^{n}K_j rW(r)\,\delta(r-r_j) \tag{2}$$

where $\Omega^4=\dfrac{\bar{\rho}h\omega^2}{D}$ and $K_j=\dfrac{k_j}{D}$. Considering an annular plate with one edge free ($r = a$) and another one clamped ($r = b$) we have the following boundary conditions (Fig. 1):

- for free edge

$$\left.\left(\frac{d^2W(r)}{dr^2}+\nu\frac{1}{r}\frac{dW(r)}{dr}\right)\right|_{r=a}=0,\quad \left.\left\{\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dW(r)}{dr}\right)\right]\right\}\right|_{r=a}=0 \tag{3, 4}$$

- for clamped edge

$$W(r)\big|_{r=b}=0,\qquad \left.\frac{\mathrm{d}W(r)}{\mathrm{d}r}\right|_{r=b}=0 \tag{5, 6}$$

where ν is the Poisson ratio.

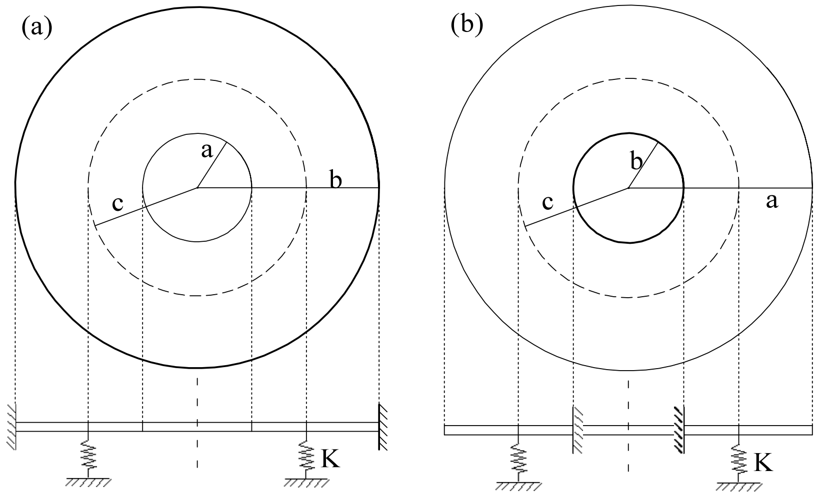

Fig. 1. Annular plate with one supporting ring: a) free-clamped, b) clamped-free

To solve this problem we use Green's function method. Using the properties of Green's function the solution to the problem can be represented in the form

$$W(r) = -\sum_{i=1}^{n} K_i \, r_i \, W(r_i) G(r, r_i) \tag{7}$$

where $G(r, \rho)$ is Green's function. Substituting $r = r_j$, $j = 1,\ldots, n$, into (7) we obtain a set of equations, which can be written in the matrix form

$$\boldsymbol{A}\boldsymbol{W} = \boldsymbol{0} \tag{8}$$

where $\boldsymbol{A} = \lfloor a_{ij} \rfloor_{1 \le i, j \le n}$, $\boldsymbol{W} = [W(r_1) \ldots W(r_n)]^T$ and $a_{ij} = K_i \, G(r_i, r_j) + \delta_{ij}$, where δ_{ij} is the Kronecker's delta. A nontrivial solution of equation (8) exists only if

$$\det \boldsymbol{A} = 0 \tag{9}$$

and this is a characteristic equation of the problem. This equation is then solved numerically with respect to the natural frequencies parameter Ω.

2. Green's functions

We use Green's function G to obtain the solution to a problem (2) with boundary conditions (3)-(6). This function satisfies the equation below

$$\frac{\mathrm{d}}{\mathrm{d}r}\left\{ r \frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r \frac{\mathrm{d}}{\mathrm{d}r} G(r,\rho)\right)\right]\right\} - r\Omega^4 G(r,\rho) = \delta(r-\rho) \tag{10}$$

Function G can be represented as a sum

$$G(r,\rho)=G_0(r,\rho)+G_1(r,\rho)H(r-\rho) \tag{11}$$

Function $G_0(r, \rho)$ is the general solution of the homogeneous equation (10) and function $G_1(r, \rho)H(r-\rho)$ is the particular solution of equation (10). Simultaneously the function G_1 satisfies the following conditions

$$G_1\Big|_{r=\rho}=\frac{\partial G_1}{\partial r}\Big|_{r=\rho}=\frac{\partial^2 G_1}{\partial r^2}\Big|_{r=\rho}=0 \text{ and } \frac{\partial^3 G_1}{\partial r^3}\Big|_{r=\rho}=\frac{1}{\rho} \tag{12}$$

It may be proved that the function G_1 is a solution of the homogeneous equation (10). The general solution of this equation has the form

$$G_1(r,\rho)=c_1J_0(r\,\Omega)+c_2I_0(r\,\Omega)+c_3Y_0(r\,\Omega)+c_4K_0(r\,\Omega) \tag{13}$$

Constants c_1, c_2, c_3, c_4 are calculated with the use of conditions (12). Having constants c_i, function G_1 can be written in the form

$$\begin{aligned}G_1(r,\rho)=\frac{1}{2\Omega^2}\big(&I_0(r\,\Omega)K_0(\rho\,\Omega)-I_0(\rho\,\Omega)K_0(r\,\Omega)+\\&+\frac{\pi}{2}\big(J_0(r\,\Omega)Y_0(\rho\,\Omega)-J_0(\rho\,\Omega)Y_0(r\,\Omega)\big)\big)\end{aligned} \tag{14}$$

Function G_0 is also the solution of homogeneous equation (10) so it can be represented in the form

$$G_0(r,\rho)=C_1J_0(r\,\Omega)+C_2I_0(r\,\Omega)+C_3Y_0(r\,\Omega)+C_4K_0(r\,\Omega) \tag{15}$$

Constants C_i are determined by taking into account (14), (15) in (11) and by using the boundary conditions (3)-(6). It is also assumed that $a > b$. To present function G_0 in a simple form, the following functions are introduced:

$$\Phi_1(z)=z\big(J_0(z)I_1(z)+J_1(z)I_0(z)\big)-2(1-\nu)J_1(z)I_1(z)$$

$$\Phi_2(z)=-z\big(J_0(z)K_1(z)-J_1(z)K_0(z)\big)+2(1-\nu)J_1(z)K_1(z)$$

$$\Phi_3(z)=z\big(Y_0(z)I_1(z)+Y_1(z)I_0(z)\big)-2(1-\nu)Y_1(z)I_1(z)$$

$$\Phi_4(z)=-z\big(Y_0(z)K_1(z)-Y_1(z)K_0(z)\big)+2(1-\nu)Y_1(z)K_1(z)$$

$$\Psi_1(u,z)=-\big(J_0(z\,\Omega)+K_0(z\,\Omega)\Phi_1(u\,\Omega)-I_0(z\,\Omega)\Phi_2(u\,\Omega)\big)$$

$$\Psi_2(u,z)=-\big(Y_0(z\,\Omega)+K_0(z\,\Omega)\Phi_3(u\,\Omega)-I_0(z\,\Omega)\Phi_4(u\,\Omega)\big)$$

$$\Psi_3(u,z) = I_0(z\Omega) + \frac{\pi}{2}\left(Y_0(z\Omega)\Phi_1(u\Omega) - J_0(z\Omega)\Phi_3(u\Omega)\right)$$

$$\Psi_4(u,z) = K_0(z\Omega) + \frac{\pi}{2}\left(Y_0(z\Omega)\Phi_2(u\Omega) - J_0(z\Omega)\Phi_4(u\Omega)\right)$$

and

$$d = Y_1(b\Omega)\Psi_1(a,b) - J_1(b\Omega)\Psi_2(a,b) + \frac{2}{\pi}\left(K_1(b\Omega)\Psi_3(a,b) + I_1(b\Omega)\Psi_4(a,b)\right)$$

The function G_0 can be written in the form:

$$\begin{aligned} G_0(r,\rho) = {} & \frac{1}{b\Omega^3 d}\cdot \\ & \left(\left(J_0(r\Omega) - (1-\nu)J_1(b\Omega)\left(I_0(r\Omega)K_1(b\Omega) + I_1(b\Omega)K_0(r\Omega)\right) + \frac{1}{2}\Psi_1(b,r)\right)\right. \\ & \Psi_2(a,\rho) - \\ & -\left(Y_0(r\Omega) - (1-\nu)Y_1(b\Omega)\left(I_0(r\Omega)K_1(b\Omega) + I_1(b\Omega)K_0(r\Omega)\right) + \frac{1}{2}\Psi_2(b,r)\right) \\ & \Psi_1(a,\rho) - \\ & -\left(\frac{2}{\pi}I_0(r\Omega) - (1-\nu)I_1(b\Omega)\left(Y_0(r\Omega)J_1(b\Omega) - Y_1(b\Omega)J_0(r\Omega)\right) - \frac{1}{\pi}\Psi_3(b,r)\right) \\ & \Psi_4(a,\rho) + \\ & \left(\frac{2}{\pi}K_0(r\Omega) + (1-\nu)K_1(b\Omega)\left(Y_0(r\Omega)J_1(b\Omega) - Y_1(b\Omega)J_0(r\Omega)\right) - \frac{1}{\pi}\Psi_4(b,r)\right) \\ & \left.\Psi_3(a,\rho)\right) \end{aligned} \tag{16}$$

Function G_0 has been derived with an assumption that $a > b$. So Green's function for the C-F annular plate (clamped inside and free outside) is given by (11). We would like to present Green's function g for a F-C annular plate (free inside and clamped outside) using the above functions G_0 and G_1.

To find function g we present it in a form

$$g(r,\rho) = g_0(r,\rho) + G_1(r,\rho)H(r-\rho) \tag{11'}$$

and assume that $a < b$. Function g_0 is the solution of homogeneous equation (10) and it must be such a function that g satisfies the boundary conditions (3)-(6). Considering the procedure of determining function G_0 and g_0 we arrived at the conclusion that $g_0 = -G_0 - G_1$. Finally, Green's function for the plate free inside and clamped outside has the form

$$g(r,\rho) = -G_0(r,\rho) - G_1(r,\rho) + G_1(r,\rho)H(r-\rho) \tag{17}$$

3. Numerical example

We present some numerical examples to demonstrate Green's function method. Numerical calculations deal with free vibrations of annular plates with one supporting elastic ring. In this case the frequency equation (9) takes the form

$$G(c,c;\Omega)+\frac{1}{cK}=0 \tag{18}$$

where c and K are the radius and stiffness coefficient of the elastic ring support, respectively. Frequency parameter values Ω_i for the first two modes of axisymmetric vibration of annular plates with an elastic ring are shown in Figure 2 as functions of the ratio $\xi = (c - b)/(a - b)$ (C-F plate) and Figure 3 as functions of the ratio $\xi = (c - a)/(b - a)$ (F-C plate), for various values of the stiffness coefficient K.

In case of free-clamped annular plate, taking the inner radius very small, obtained frequency parameter Ω is in good agreement with a frequency parameter obtained for a clamped circular plate with all parameters (thickness, bending rigidity, mass per unit volume) and outer radius the same as for annular plate (see Table 1).

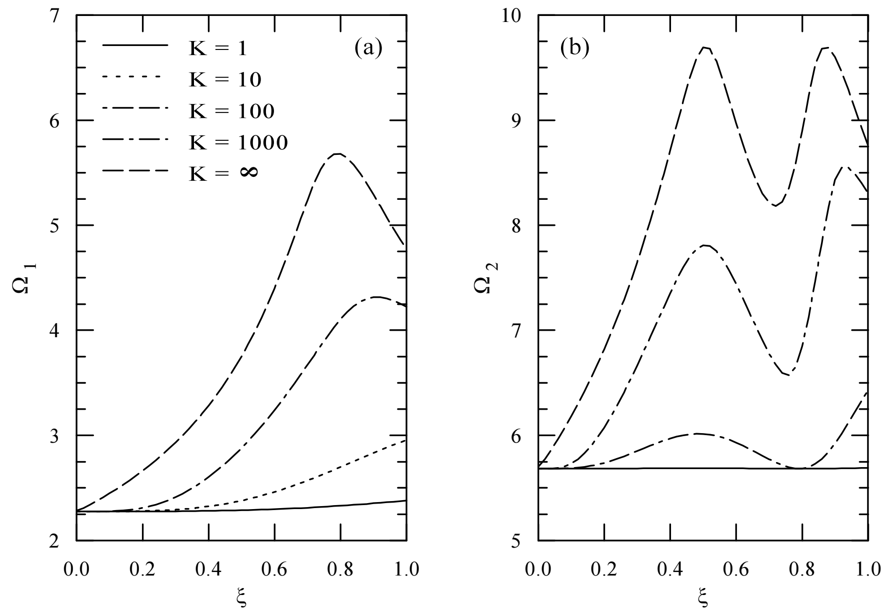

Fig. 2. Frequency parameter values Ω_i for the first two modes of axisymmetric vibration of clamped-free annular plate with an supporting elastic ring as a functions of the ratio $\xi = (c - b)/(a - b)$ for various values of the stiffness coefficient K; $b/a = 0.2$

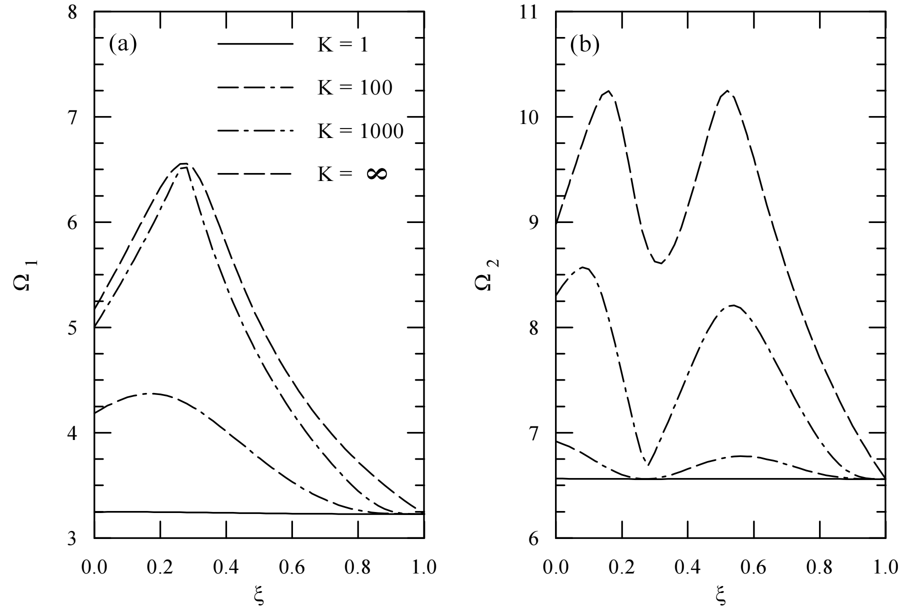

Fig. 3. As Fig. 2 but for free-clamped plate, $\xi = (c - a) / (b - a)$, $a/b = 0.2$

Table 1

Non-dimensional fundamental frequency parameter *Ω* for free-clamped annular plate (*a* - inner radius, *b* - outer radius) and clamped circular plate; *c* - radius of the rigid supporting ring

	a/b	$K = 0$[1)]	$K = \infty$		
			c/*b* = 0.3	*c*/*b* = 0.5	*c*/*b* = 0.7
annular plate	0.05	3.19201637	5.98017232	5.51569022	4.21603894
	0.01	3.19601127	5.99784749	5.53297973	4.22450863
	0.005	3.19616776	5.99850109	5.53390013	4.22488759
	0.001	3.19621849	5.99871254	5.53420364	4.22501139
	0.0005	3.19622009	5.99871917	5.53421321	4.22501528
	0.0001	3.19622060	5.99872129	5.53421628	4.22501653
circular plate		3.19622062	5.99872138	5.53421640	4.22501658

1) a plate without supporting ring

All calculations were performed for the Poisson ratio $\nu = 0.3$.
The results have shown that both the radius and the stiffness coefficient of the support have a significant effect on free vibration frequencies of the plate.

Conclusions

Green's function method has been applied to obtain the exact solution to the axisymmetric vibration problem of a uniform annular plate supported by elastic concentric rings. For free-clamped and clamped-free annular plates Green's functions were derived. Numerical examples deal with vibrations of plates with one supporting ring.

However, the solution can be used to analyze the vibrations of any annular plates (free-clamped or clamped-free) with an arbitrary number of ring supports.

Moreover, the solution for free-clamped annular plate can be used to vibration analysis of the circular clamped plate (by taking small inner radius).

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