Please cite this article as:
Stanisław Kukla, Agnieszka Owczarek, Stochastic vibration of a Bernoulli-Euler beam under random excitation, Scientific Research of the Institute of Mathematics and Computer Science, 2005, Volume 4, Issue 1, pages 71-78.
The website: http://www.amcm.pcz.pl/

STOCHASTIC VIBRATION OF A BERNOULLI-EULER BEAM UNDER RANDOM EXCITATION

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Abstract. In this paper the problem of randomly excited vibration of a Bernoulli-Euler beam with an elastic support is considered. The pointwise, stationary random in time force effects on the beam in a fixed point, exciting its transverse vibration. The statistical properties of the response are described in terms of covariance of the random excitation. The effect of position of the random force as well the rigidity of the elastic support on the standard deviation of the beam deflection has been numerically investigated.

Introduction

The problems of vibrations of beams subjected to random excitations are of great practical importance. Beams are elements of many engineering structures (for instance machines, buildings, bridges), and their random excitations can cause fluid pressure, earthquake loads, moving or impact loads. Mathematical description of transverse vibrations of the beams based on Bernoulli-Euler or Rayleigh beam theories establish the fourth order partial differential equations, which are completed by suitable initial and boundary conditions.

The vibration analysis of the beams excited by random loads are the subject of papers [1-4]. In reference [1], Dahlberg uses the modal analysis technique to investigate the influence of modal cross-spectral densities on the spectral densities of some responses of simply supported beams. The random response of damped beams was studied by Jacquot in reference [2]. The author presents a method of vibration analysis using the response power spectral density function and mean-square response of considered beam structures excited by a second stationary random process. In paper [3] Kukla and Skalmierski dealt with the random vibration of a clamped-pinned beam. The flux of energy which is emitted by the vibrating beam was investigated. Papadimitriou et al. in work [4] provide a methodology for optimal establishment of the number and location of sensors on randomly vibrating structures for the purpose of the response predictions at unmeasured locations in structural systems. The authors referees the results of considerations to randomly vibrating beams and plates.

In the present paper, the transverse vibrations of a beam induced by a random, pointwise force, are analysed. The excitation force is assumed in separable form as a stationary random in time process. The variance of the random beam deflection has been derived. The analytical solution was used in numerical investigations of the beam vibration excited by white noise process.

1. Formulation and solution to the problem

Consider a beam randomly loaded by a transverse force $p(x,t)$. The vibration of the beam is governed by the differential equation

$$EI\frac{\partial^4 w}{\partial x^4}+c\frac{\partial w}{\partial t}+m\frac{\partial^2 w}{\partial t^2}=p(x,t) \qquad (1)$$

where c is the damping coefficient, m is the mass per unit length of the beam, E is Young's modulus of elasticity, I is the moment of inertia of the beam cross-section, and $w(x,t)$ represents the beam's deflection at the cross-section x at time t.

At each end of the finite beam two conditions must be satisfied, which may be written symbolically in the following form

$$\mathbf{B}_0\left[w(x,t)\right]\big|_{x=0}=0,\quad \mathbf{B}_1\left[w(x,t)\right]\big|_{x=L}=0 \qquad (2)$$

where $\mathbf{B}_0$ and $\mathbf{B}_1$ are linear, spatial, two dimensional differential operators and L denotes the length of the beam.

The solution of the problem is searched in form of a sum

$$w(x,t)=\sum_{k=1}^{\infty}y_k(x)\phi_k(t) \qquad (3)$$

where functions $y_k(x)$, $k = 1, 2,\ldots$, satisfied the differential equation

$$EI\frac{d^4 y_k(x)}{dx^4}-\omega_k^2\, m\, y_k(x)=0 \qquad (4)$$

and the following boundary conditions:

$$\overline{\mathbf{B}}_0\left[y_k(x)\right]\big|_{x=0}=0,\quad \overline{\mathbf{B}}_1\left[y_k(x)\right]\big|_{x=L}=0 \quad \text{for } k=1,2,... \qquad (5)$$

The functions $y_k(x)$ as solutions of the eigenproblem (4, 5), create an orthogonal system of functions in interval $[0,L]$. It means that

$$\int_0^L y_k(x)\,y_l(x)\,dx=d_k\,\delta_{kl} \qquad (6)$$

where δ_{kl} is the Kronecker delta and $d_k = \int_0^L [y_k(x)]^2 \, dx$.

Substituting the function $w(x,t)$ in the form given by equation (3) into differential equation (1), multiplying the obtained equation by the function $y_l(x)$ for $l = 1, 2,\ldots$, after integrating both sides of the equation in limits 0, L, and using (6), the following equations are obtained:

$$\ddot{\phi}_k(t) + 2\zeta_k \omega_k \dot{\phi}_k(t) + \omega_k^2 \phi_k(t) = \frac{1}{m d_k} \tilde{p}_k(t), \quad \text{k} = 1, 2,\ldots \tag{7}$$

where $\zeta_k = \dfrac{c}{2m\omega_k}$, and

$$\tilde{p}_k(t) = \int_0^L p(x,t) y_k(x) dx \tag{8}$$

Note that $\tilde{p}_k(t) = 0$ for $t < 0$, since $p(x,t) = 0$ for $t < 0$.

The solution of the differential equations (7) with zero initial conditions and $0 \le \zeta_k < 1$, can be expressed by Duhamel's integral as

$$\phi_k(t) = \int_{-\infty}^{+\infty} G_k(t-\tau) \tilde{p}_k(\tau) d\tau \tag{9}$$

where $G_k(t)$ is a Green's function which is given by [5]:

$$G_k(t) = \begin{cases} \dfrac{1}{m d_k \omega_{dk}} e^{-\zeta_k \omega_k t} \sin \omega_{dk} t & \text{for } t \ge 0 \\ 0 & \text{for } t < 0 \end{cases} \tag{10}$$

with $\omega_{dk} = \omega_k \sqrt{1-\zeta_k^2}$. Finally, on the basis of equations (3) and (9), one obtains

$$w(x,t) = \sum_{k=1}^{\infty} y_k(x) \int_{-\infty}^{+\infty} G_k(t-\tau) \tilde{p}_k(\tau) d\tau \tag{11}$$

The statistical properties of the output signal can be described in terms of covariance $C_{ww}(x_1, x_2, t_1, t_2)$. In light of equation (3), the function is given by

$$C_{ww}(x_1, x_2, t_1, t_2) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} y_j(x_1) y_k(x_2) C_{\phi_j \phi_k}(t_1, t_2) \tag{12}$$

where $C_{\phi_j \phi_k}(t_1,t_2)=E[\phi_j(t_1)\phi_k(t_2)]$ and $E[\cdot]$ denotes mathematical expectation. Introducing the function: $\widetilde{C}_{ww}(t_1,t_2)=\int_0^L C_{ww}(x,x,t_1,t_2)dx$ and taking into account equations (6) and (12), one obtains

$$\widetilde{C}_{ww}(t_1,t_2)=\sum_{k=1}^{\infty} d_k\, C_{\phi_k \phi_k}(t_1,t_2) \tag{13}$$

where the covariance $C_{\phi_k \phi_k}(t_1,t_2)$, reads [6]

$$C_{\phi_k \phi_k}(t_1,t_2)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G_k(t_1-\tau_1)G_k(t_2-\tau_2)C_{\widetilde{p}_k \widetilde{p}_k}(\tau_1,\tau_2)d\tau_1\, d\tau_2 \tag{14}$$

If excitation force may be written in the separable form: $p(x,t) = P(x)\ \varphi(t)$, where $\varphi(t)$ is a random function, then the covariance $C_{\widetilde{p}_k \widetilde{p}_k}(\tau_1,\tau_2)$ evaluated on the basis of equation (8) is given as

$$C_{\widetilde{p}_k \widetilde{p}_k}(\tau_1,\tau_2)=b_k C_{\varphi\varphi}(\tau_1,\tau_2) \tag{15}$$

where $b_k = \left[\int_0^L P(x)\, y_k(x)dx\right]^2$.

Assume that the input signal is a stationary process, i.e. the following condition is satisfied: $C_{\widetilde{p}_k \widetilde{p}_k}(\tau_1,\tau_2)=C_{\widetilde{p}_k}(\tau_1-\tau_2)$. In this case, for the process given by equation (9), the following relationship holds [7]:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G_k(t-\tau_1)\overline{G}_k(t-\tau_2)C_{\widetilde{p}_k}(\tau_1-\tau_2)d\tau_1\, d\tau_2 = \int_{-\infty}^{+\infty}\left|\widehat{G}_k(i\omega)\right|^2 S_{\widetilde{p}_k}(\omega)d\omega \tag{16}$$

where the over bar denotes complex conjugate, the function $\widehat{G}_k(i\omega)$ is a Fourier transform of $G_k(\tau)$ i.e.: $\widehat{G}_k(i\omega)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} G_k(\tau)e^{-i\omega\tau}d\tau$. Moreover, the function $S_{\widetilde{p}_k}(\omega)$ is the spectral density of the process $\widetilde{p}_k(\tau)$ and form with the correlation function $C_{\widetilde{p}_k}(\tau)$ a Fourier transform pair [8]:

$$S_{\widetilde{p}_k}(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} C_{\widetilde{p}_k}(\tau)e^{-i\omega\tau}\, d\tau,\quad C_{\widetilde{p}_k}(\tau)=\int_{-\infty}^{\infty} S_{\widetilde{p}_k}(\omega)e^{i\omega\tau}\, d\omega \tag{17}$$

In case of separable excitation force, using equation (15), the spectral density $S_{\tilde{p}_k}(\omega)$ of the stationary process becomes

$$S_{\tilde{p}_k}(\omega) = b_k S_\varphi(\omega) \tag{18}$$

For the white noise excitation, the spectral density $S_\varphi(\omega)$ takes the form: $S_\varphi(\omega) = S_0$, where S_0 is the constant value.

The variance of the output process for the white noise input process, on the basis of equations (13), (16), (18), may be expressed by

$$Var(t) = \tilde{C}_{ww}(t,t) = S_0 \sum_{k=1}^{\infty} b_k\, d_k \int_{-\infty}^{+\infty} \left|\hat{G}_k(\omega)\right|^2 d\omega \tag{19}$$

The equation (19) shows that the variance is not dependent from t: $Var(t) = \sigma^2 = \text{const}$.

If the excitation force $p(x,t) = P(x)\ \varphi(t)$ is stationary random in time t and effect pointwise on the beam at $x = x_0$, then assumes: $P(x) = P_0\, \delta(x - x_0)$, where $\delta(\cdot)$ is Dirac delta function. In this case the coefficients b_k occurring in equation (19), are given in the form: $b_k = P_0^2\, y_k^2(x_0)$, k = 1, 2,... . Moreover, the Fourier transform $\hat{G}_k(\omega)$ of the function $G_k(t)$ given by equation (10), can be written in the form

$$\hat{G}_k(\omega) = \frac{1}{m\, d_k} \frac{1}{\omega_{dk}^2 + (\zeta_k \omega_k + i\omega)^2} \tag{20}$$

Finally, after evaluation of the integral which occurs in equation (19), the variance σ^2 becomes

$$\sigma^2 = \frac{\pi P_0^2 S_0}{c\, m} \sum_{k=1}^{\infty} \frac{y_k^2(x_0)}{d_k \omega_k^2} \tag{21}$$

The coefficients d_k can be determined when the eigenfunctions $y_k(x)$ of the problem (4), (5) are known.

2. Numerical example

Consider the random vibration of a simply supported beam. The boundary conditions, the eigenfunctions $y_k(x)$ and the eigenfrequencies ω_k corresponding to the beam are:

$$y_k\big|_{x=0} = y_k''\big|_{x=0} = 0\,, \quad y_k\big|_{x=L} = y_k''\big|_{x=L} = 0 \tag{22}$$

$$y_k(x) = \sin\frac{\pi k x}{L}, \quad \omega_k = \left(\frac{\pi k}{L}\right)^2 \sqrt{\frac{EI}{m}} \quad \text{for } k = 1, 2, \ldots \tag{23}$$

The coefficients d_k are given by: $d_k = \frac{L}{2}$, and on the basis of (21) becomes

$$\sigma^2 = \frac{2 P_0^2 S_p}{c\,EI} \left(\frac{L}{\pi}\right)^3 \sum_{k=1}^{\infty} \frac{1}{k^4} \sin\frac{\pi k x_0}{L} \tag{24}$$

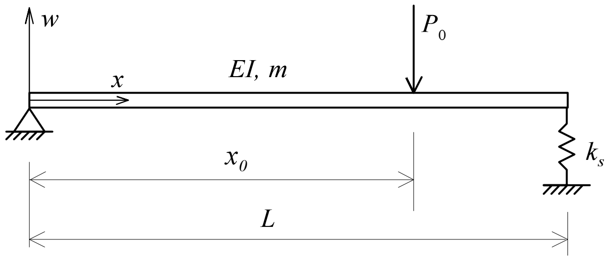

Fig. 1. A sketch of the pinned-free beam with a tip elastic support

The second example concerns the beam of which the left end ($x = 0$) is simply supported (pinned) and the right ($x = L$) is supported by a translational spring with the spring constant k_s. The boundary conditions for $x = 0$ are the same as in the first example (equation (22a)), and for $x = L$ there are as follows:

$$\left.\frac{\partial^2 w}{\partial x^2}\right|_{x=L} = 0, \quad EI\left.\frac{\partial^3 w}{\partial x^3}\right|_{x=L} - k_s \left.w\right|_{x=L} = 0 \tag{25}$$

The eigenfunctions can be written in the form:

$$y_k(x) = \sinh\Omega_k \sin\frac{\Omega_k x}{L} + \sin\Omega_k \sinh\frac{\Omega_k x}{L}, \quad k = 1, 2, \ldots \tag{26}$$

where $\omega_k = \Omega_k^2 \sqrt{\frac{EI}{mL^4}}$ and the values of Ω_k are roots of the equation

$$\Omega_k^3 \left(\cos\Omega_k \sinh\Omega_k - \sin\Omega_k \cosh\Omega_k\right) + 2K_s \sin\Omega_k \sinh\Omega_k = 0 \tag{27}$$

The coefficients $d_k = \int_0^L [y_k(x)]^2\, dx$ are given by

$$d_k = \frac{L}{4}\left[\cos 2\Omega_k + \cosh 2\Omega_k + \frac{12 K_s}{\Omega_k^4} \sin^2\Omega_k \sinh^2\Omega_k - 2\right] \tag{28}$$

with $K_s = \dfrac{k_s L^3}{EI}$. Assuming, similarly as in the case of simply supported beam, that excitation force effects pointwise on the beam at $x = x_0$, the variance of the process on the basis of equation (21) in the following form can be written:

$$\sigma^2 = \frac{2\pi P_0^2 S_p L^3}{c\,EI} \sum_{k=1}^{\infty} \frac{y_k^2(x_0)}{\overline{d}_k} \tag{29}$$

where $\overline{d}_k = \Omega_k^4 \left[\sinh^2 \Omega_k - \sin^2 \Omega_k\right] + 6 K_s \sin^2 \Omega_k \sinh^2 \Omega_k$.

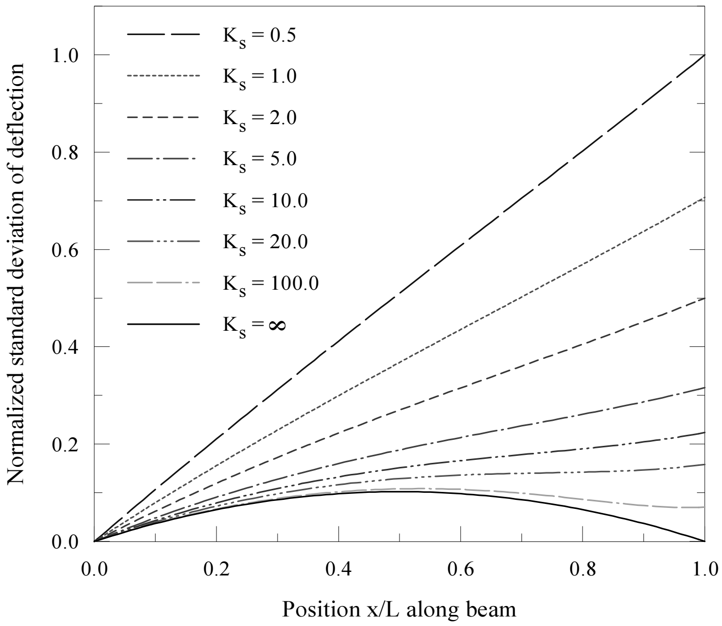

Fig. 2. Normalized standard deviation (σ/σ_0 of the beam deflection versus position (x/L) of poinwise stationary random excitation force

The normalized standard deviation of the random vibration of the beam with one end simply supported and the other free using equation (29), has been numerically evaluated. It is assumed that the beam at the free end is elastically supported and subjected at the position x_0/L to a pointwise, stationary random excitation force. The calculations were performed for various values of the coefficient K_s characterizing the spring constant. The standard deviation was normalized with respect to the factor calculated as the standard deviation of the beam deflection with non-dimensional spring constant $K_s = 0.5$, when the force effects at $x_0/L = 1$. The normalized standard deviations as functions of position of the excitation force are presented in Figure 2. The solid line was plotted with assumption that the K_s

tends to infinity, i.e. the case of simply supported beam is obtained. In this case, the standard deviation of beam deflection can be obtained as the square root of variance given by equation (24). The curves in Figure 2 show that value of the standard deviation can increases several times when the coefficient K_s decreases.

Conclusions

In this paper, an investigation of the random vibration of a Bernoulli-Euler beam with an elastic support is presented. Solution of the problem is obtained in analytical form. The covariance of the response is expressed by the covariance of excitation process. In numerical calculations it is assumed that the beam vibration is excited by white noise process. The results show that the position of the random force as well the rigidity of the elastic support significantly effect on the standard deviation of the beam deflection. Although the example presented concern the simply supported - free beam with elastic support at free end, the approach can be applied to beams with another end constraints and discrete attachments.

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