USING FRACTIONAL CALCULUS FOR GENERATION OF α-STABLE LÉVY PROBABILITY DENSITY FUNCTIONS

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Abstract. This paper deals with numerical solutions of a partial differential equation of fractional order. This type of equation describes a process of anomalous diffusion. In this equation a space fractional derivative known as the Riesz operator occurs. We propose a new approach of discretization of the Riesz operator. Using this approach we present a method which utilises this discretization to the solution of equation based on the finite difference method. We also present fundamental solutions of this equation in one-dimensional space which are so-called the class of Lévy stable densities of index α. In the final part of this paper some numerical examples are shown.

Introduction

The process of diffusion is a transport process of matter from one part of a system to another. This results from random molecular motions. Such transport can be characterised by the mean-square displacement of particle positions $< x^2(t) >$. In the classical diffusion a mean-square displacement grows linearly with time $< x^2(t) > = 2K_2 t$, where K_2 is a coefficient of diffusion m^2/s. For modelling of such description the Fick's law is used. Examples of the classical diffusion are Brownian motion of particles and heat transfer. Transfer in the complex and non-homogenous background is related to some deviations from the standard Fick's law and it is so-called *anomalous diffusion* [1-4]. It is characterised by the occurrence of a mean-square displacement of the form $< x^2(t) > \sim t^\alpha$, for $0 < \alpha < 2$, or the second moment does not exist $< x^2(t) > \to \infty$. This second case is characterized by rare but extremely large jumps of diffusion particle - so-called the Lévy motion or Lévy flights [1, 4, 5].

The fundamental solution for the Cauchy problem of the classical linear diffusion equation can be interpreted as Gaussian spatial probability density function evolving in time. In the last years have grown number of papers which provide mathematical models based on equations with fractional derivatives [6-8] for describing phenomena of anomalous diffusion [4, 9, 10]. In this work we will consider an equation of anomalous diffusion with space fractional derivative which generate a class of (non-Gaussian) symmetric Lévy stable densities of index α. In [5, 11] the reader can find more details on Lévy stable densities.

1. Mathematical model

In this paper we consider a partial differential equation of fractional order (known as anomalous diffusion equation [9, 12, 13]) in which, in comparison to the classical diffusion equation, the second-order space derivative, is replaced by the derivative of fractional order α. This equation has the following form

$$\frac{\partial}{\partial t}P(x,t)=K_{\alpha}\frac{\partial^{\alpha}}{\partial|x|^{\alpha}}P(x,t), \qquad t\geq 0, \quad -\infty<x<\infty \tag{1}$$

where $P(x,t)$ is the probability density function, $\partial^{\alpha}/\partial|x|^{\alpha}\,P(x,t)$ is the Riesz fractional operator [8, 12], α is the real order of this operator, K_{α} is the coefficient of generalized (anomalous) diffusion with the unit of measure m^{α}/s. According to [8, 12] the Riesz fractional operator for $0<\alpha\leq 2$, $\alpha\neq 1$, and for the one-variable function $u(x)$ is defined as

$$\frac{\partial^{\alpha}}{\partial|x|^{\alpha}}u(x)=\begin{cases}-\dfrac{1}{2\cos\dfrac{\pi\alpha}{2}}\left({}_{-\infty}D_x^{\alpha}u(x)+{}_{x}D_{\infty}^{\alpha}u(x)\right), & \text{for } \alpha\neq 1\\ -\dfrac{d}{dx}H\,u(x), & \text{for } \alpha=1\end{cases} \tag{2}$$

where ${}_{-\infty}D_x^{\alpha}u(x)$ and ${}_{x}D_{\infty}^{\alpha}u(x)$ are fractional derivatives in the Riemann-Liouville sense and $Hu(x)$ is the Hilbert transform defined as:

$${}_{-\infty}D_x^{\alpha}u(x)=\begin{cases}\dfrac{d}{dx}\left[{}_{-\infty}I_x^{1-\alpha}u(x)\right], & \text{for } 0<\alpha\leq 1\\ \dfrac{d^2}{dx^2}\left[{}_{-\infty}I_x^{2-\alpha}u(x)\right], & \text{for } 1<\alpha\leq 2\end{cases} \tag{3}$$

$${}_{x}D_{\infty}^{\alpha}u(x)=\begin{cases}-\dfrac{d}{dx}\left[{}_{x}I_{\infty}^{1-\alpha}u(x)\right], & \text{for } 0<\alpha\leq 1\\ \dfrac{d^2}{dx^2}\left[{}_{x}I_{\infty}^{2-\alpha}u(x)\right], & \text{for } 1<\alpha\leq 2\end{cases} \tag{4}$$

$$H\,u(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{u(\xi)}{x-\xi}d\xi \tag{5}$$

Occurring in (3) and (4) fractional operators of order α: ${}_{-\infty}I_x^{\alpha}u(x)$ and ${}_{x}I_{\infty}^{\alpha}u(x)$ are the left- and right-side of Weyl fractional integrals [7, 8] which definitions are:

$$ {}_{-\infty}I_x^{\alpha}u(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} \frac{u(\xi)}{(x-\xi)^{1-\alpha}} d\xi \tag{6} $$

$$ {}_{x}I_{\infty}^{\alpha}u(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} \frac{u(\xi)}{(\xi-x)^{1-\alpha}} d\xi \tag{7} $$

Assuming $\alpha = 2$ in equation (1) we obtain classical diffusion, i.e. so-called the heat transfer equation, but for $\alpha = 1$: $d/d|x|\ u(x) \neq d/dx\ u(x)$. For analytic solution of equation (1) we can apply Fox's H-function [4], but we numerically solve this equation when additional non-linear term may occur. We know solutions in the simple form only for $\alpha = 2$ and $\alpha = 1$ as Gaussian $P_2(x,t)$ and Cauchy $P_1(x,t)$ pdf's

$$ P_2(x,t) = \frac{1}{2\sqrt{\pi K_2 t}} \exp\left(\frac{-x^2}{4K_2 t}\right), \qquad P_1(x,t) = \frac{1}{\pi} \frac{K_1 t}{(K_1 t)^2 + x^2} \tag{8} $$

Some numerical methods used in solution of fractional differential equations can be found in [12]. However they are applied to the infinity domain. In this work we propose a discretization scheme for the Riesz derivative and we consider equation (1) in one-dimensional domain Ω: $-L \leq x \leq L$ (we omit the case when $\alpha = 1$) with absorbing boundaries (Dirichlet conditions)

$$ P(x,t)\big|_{x=-L} = P(x,t)\big|_{x=L} = 0 \tag{9} $$

and with the initial condition

$$ P(x,t)\big|_{t=0} = c_0(x) \tag{10} $$

2. Numerical method

According to the finite difference method [14, 15] we consider a discrete from of equation (1) both in time and space. In the previous work [16] we solved numerically the fractional partial differential equation similar to (1) where only the time-fractional derivative has been taken into account. We called this method FFDM (the Fractional FDM). Extending our considerations we can say that solution of equation (1) needs proper approximation of the Riesz derivative (2) in some numerical schemes.

Here we introduce another definition of the fractional derivative in the Caputo form [7, 9] as:

$$ \begin{aligned} {}_{-\infty}^{\ C}D_x^{\alpha}u(x) &= {}_{-\infty}I_x^{m-\alpha}u^{(m)}(x) \\ {}_{x}^{C}D_{\infty}^{\alpha}u(x) &= {}_{x}I_{\infty}^{m-\alpha}u^{(m)}(x) \end{aligned} \tag{11} $$

where $m \in \mathbf{N}$, $m-1 < \alpha \le m$, and $u^{(m)}$ are first and second derivatives, for $m = 1, 2$. Above definitions base on assumption in [13] that expressions (3), (4) and (11) are the same when the lower/upper limit of integration tends to minus/plus infinity. Thus we have

$$ {}_{-\infty}D_x^{\alpha}u(x) = {}_{-\infty}^{\;\;C}D_x^{\alpha}u(x) \quad \text{and} \quad {}_xD_{\infty}^{\alpha}u(x) = {}_x^CD_{\infty}^{\alpha}u(x) \tag{12}$$

2.1. Approximation of the Riesz derivative

We start numerical analysis from discretization of operators (11) related to definition (2). Therefore we introduce a homogenous spatial grid $-\infty < \ldots < x_{i-2} < x_{i-1} < x_i < x_{i+1} < x_{i+2} < \ldots < \infty$ with the step $h = x_i - x_{i-1}$. We denote a value of function $u(x)$ in the point x_i as $u_i = u(x_i)$, for $i \in \mathbf{Z}$. According to changes in the parameter α in (2) we distinguish two cases of discrete approximation of the Riesz derivative.

The first case includes changes in the parameter α for the range $0 < \alpha < 1$. We rewrite operator (2) using the Caputo definition as

$$\begin{aligned} \left.\frac{\partial^{\alpha}}{\partial|x|^{\alpha}}u(x)\right|_{x=x_i} &= \frac{-1}{2\cos\frac{\pi\alpha}{2}}\left[\frac{1}{\Gamma(1-\alpha)}\int_{-\infty}^{x_i}\frac{u'(\xi)}{(x_i-\xi)^{\alpha}}d\xi - \frac{1}{\Gamma(1-\alpha)}\int_{x_i}^{\infty}\frac{u'(\xi)}{(\xi-x_i)^{\alpha}}d\xi\right] \\ &= \frac{-1}{2\Gamma(1-\alpha)\cos\frac{\pi\alpha}{2}}\left[\sum_{k=0}^{\infty}\int_{x_{i-k-1}}^{x_{i-k}}\frac{u'(\xi)}{(x_i-\xi)^{\alpha}}d\xi - \sum_{k=0}^{\infty}\int_{x_{i+k}}^{x_{i+k+1}}\frac{u'(\xi)}{(\xi-x_i)^{\alpha}}d\xi\right] \\ &\approx \frac{-1}{2\Gamma(1-\alpha)\cos\frac{\pi\alpha}{2}}\left[\sum_{k=0}^{\infty}\tilde{u}'_{i-k}\int_{x_{i-k-1}}^{x_{i-k}}\frac{1}{(x_i-\xi)^{\alpha}}d\xi - \sum_{k=0}^{\infty}\tilde{\tilde{u}}'_{i+k}\int_{x_{i+k}}^{x_{i+k+1}}\frac{1}{(\xi-x_i)^{\alpha}}d\xi\right] \end{aligned} \tag{13}$$

where $\tilde{u}'_j$ and $\tilde{\tilde{u}}'_j$ are difference schemes which approximate the first derivative of integer order on the intervals $[x_{j-1}, x_j]$ and $[x_j, x_{j+1}]$ respectively. We propose the following weighed forms of these difference schemes as

$$\begin{aligned} \tilde{u}'_j &= \frac{1}{2}\left[\frac{u_j - u_{j-1}}{h} + \frac{(1-\alpha)(u_j - u_{j-1}) + \alpha(u_{j+1} - u_j)}{h}\right] \\ &= \frac{1}{2h}\left(\alpha u_{j+1} + 2(1-\alpha)u_j + (\alpha-2)u_{j-1}\right), \end{aligned} \tag{14}$$

$$\begin{aligned} \tilde{\tilde{u}}'_j &= \frac{1}{2}\left[\frac{u_{j+1} - u_j}{h} + \frac{(1-\alpha)(u_{j+1} - u_j) + \alpha(u_j - u_{j-1})}{h}\right] \\ &= \frac{1}{2h}\left((2-\alpha)u_{j+1} + 2(\alpha-1)u_j + (-\alpha)u_{j-1}\right) \end{aligned} \tag{15}$$

We introduce these formulae because we want to obtain various transitions from one to another difference scheme which are connected with the first derivative of integer order. For example, putting α = 1 into (14) and (15) we obtain wide known the central-difference approximation of first derivative and putting value α = 0 we get the backward- (14) or forward- (15) difference schemes respectively. In this way we would like to avoid a problem of singularity in the operator (2) for α → 1.

Denoting by

$$v_k^{(\alpha)} = \int_{x_{i-k-1}}^{x_{i-k}} \frac{1}{(x_i-\xi)^\alpha} d\xi \equiv \int_{x_{i+k}}^{x_{i+k+1}} \frac{1}{(\xi-x_i)^\alpha} d\xi = h^{1-\alpha}\frac{(k+1)^{1-\alpha}-k^{1-\alpha}}{1-\alpha} \tag{16}$$

we have

$$\begin{aligned}\left.\frac{\partial^\alpha}{\partial|x|^\alpha}u(x)\right|_{x=x_i} \approx \frac{-1}{2\Gamma(1-\alpha)\cos\frac{\pi\alpha}{2}}\Bigg[&\sum_{k=0}^{\infty}\frac{1}{2h}\left(\alpha u_{i-k+1}+2(1-\alpha)u_{i-k}+(\alpha-2)u_{i-k-1}\right)v_k^{(\alpha)} \\ &-\sum_{k=0}^{\infty}\frac{1}{2h}\left((2-\alpha)u_{i+k+1}+2(\alpha-1)u_{i+k}+(-\alpha)u_{i+k-1}\right)v_k^{(\alpha)}\Bigg]\end{aligned} \tag{17}$$

Finally we can write the discrete form of (2) for 0 < α < 1, as

$$\left.\frac{\partial^\alpha}{\partial|x|^\alpha}u(x)\right|_{x=x_i} \approx \frac{1}{h^\alpha}\sum_{k=-\infty}^{\infty}u_{i+k}w_{|k|}^{(\alpha)}, \tag{18}$$

where coefficients $w_k^{(\alpha)}$ have the following form

$$w_k^{(\alpha)} = \frac{-1}{4\Gamma(2-\alpha)\cos\frac{\pi\alpha}{2}}\begin{cases}2\left(2^{1-\alpha}\alpha-3\alpha+2\right), & \text{for } k=0\\ 3^{1-\alpha}\alpha+2^{1-\alpha}(2-3\alpha)+4\alpha-4, & \text{for } k=1\\ (k+2)^{1-\alpha}\alpha+(k+1)^{1-\alpha}(2-3\alpha) & \\ \quad +k^{1-\alpha}(3\alpha-4)+(k-1)^{1-\alpha}(2-\alpha), & \text{for } k\geq 2\end{cases} \tag{19}$$

The second case involves changes in the parameter α for the range 1 < α ≤ 2. Similar to previous calculations we write operator (2) using the Caputo form as

$$\begin{aligned}\left.\frac{\partial^\alpha}{\partial|x|^\alpha}u(x)\right|_{x=x_i} &= \frac{-1}{2\cos\frac{\pi\alpha}{2}}\left[\frac{1}{\Gamma(2-\alpha)}\int_{-\infty}^{x_i}\frac{u''(\xi)}{(x_i-\xi)^{\alpha-1}}d\xi+\frac{1}{\Gamma(2-\alpha)}\int_{x_i}^{\infty}\frac{u''(\xi)}{(\xi-x_i)^{\alpha-1}}d\xi\right] \\ &\approx \frac{-1}{2\Gamma(2-\alpha)\cos\frac{\pi\alpha}{2}}\left[\sum_{k=0}^{\infty}\tilde{u}''_{i-k}\int_{x_{i-k-1}}^{x_{i-k}}\frac{1}{(x_i-\xi)^{\alpha-1}}d\xi+\sum_{k=0}^{\infty}\tilde{\tilde{u}}''_{i+k}\int_{x_{i+k}}^{x_{i+k+1}}\frac{1}{(\xi-x_i)^{\alpha-1}}d\xi\right]\end{aligned} \tag{20}$$

where $\tilde{u}_j''$ and $\tilde{\tilde{u}}_j''$ are difference schemes for the second derivative of integer order which we approximate by the following formulae

$$\begin{aligned}\tilde{u}_j'' &= \frac{1}{2}\left[\frac{u_{j+1}-2u_j+u_{j-1}}{h^2}+\frac{(\alpha-1)(u_{j+1}-2u_j+u_{j-1})+(2-\alpha)(u_j-2u_{j-1}+u_{j-2})}{h^2}\right]\\ &= \frac{1}{2h^2}\left(\alpha u_{j+1}+(2-3\alpha)u_j+(3\alpha-4)u_{j-1}+(2-\alpha)u_{j-2}\right)\end{aligned} \tag{21}$$

$$\begin{aligned}\tilde{\tilde{u}}_j'' &= \frac{1}{2}\left[\frac{u_{j+1}-2u_j+u_{j-1}}{h^2}+\frac{(\alpha-1)(u_{j+1}-2u_j+u_{j-1})+(2-\alpha)(u_{j+2}-2u_{j+1}+u_j)}{h^2}\right]\\ &= \frac{1}{2h^2}\left((2-\alpha)u_{j+2}+(3\alpha-4)u_{j+1}+(2-3\alpha)u_j+\alpha u_{j-1}\right)\end{aligned} \tag{22}$$

We can observe that putting α = 2 into (21) and (22) we determine the classical central-difference scheme and for α = 1 we obtain the backward/forward four-point discretizations of the second derivative of integer order.

Denoting by

$$v_k^{(\alpha)} = \int_{x_{i-k-1}}^{x_{i-k}} \frac{1}{(x_i-\xi)^{\alpha-1}}d\xi \equiv \int_{x_{i+k}}^{x_{i+k+1}} \frac{1}{(\xi-x_i)^{\alpha-1}}d\xi = h^{2-\alpha}\frac{(k+1)^{2-\alpha}-k^{2-\alpha}}{2-\alpha} \tag{23}$$

we have

$$\begin{aligned}\left.\frac{\partial^\alpha}{\partial|x|^\alpha}u(x)\right|_{x=x_i} &= \frac{-1}{2\Gamma(2-\alpha)\cos\frac{\pi\alpha}{2}}\times\\ &\left[\sum_{k=0}^{\infty}\frac{1}{2h^2}\left(\alpha u_{i-k+1}+(2-3\alpha)u_{i-k}+(3\alpha-4)u_{i-k-1}+(2-\alpha)u_{i-k-2}\right)v_k^{(\alpha)}\right.\\ &\left.+\sum_{k=0}^{\infty}\frac{1}{2h^2}\left((2-\alpha)u_{i+k+2}+(3\alpha-4)u_{i+k+1}+(2-3\alpha)u_{i+k}+\alpha u_{i+k-1}\right)v_k^{(\alpha)}\right]\end{aligned} \tag{24}$$

Finally we can write the discrete form of expression (2) $1 < \alpha \le 2$, as

$$\left.\frac{\partial^\alpha}{\partial|x|^\alpha}u(x)\right|_{x=x_i} = \frac{1}{h^\alpha}\sum_{k=-\infty}^{\infty}u_{i+k}w_{|k|}^{(\alpha)} \tag{25}$$

where coefficients $w_k^{(\alpha)}$ here are

$$w_k^{(\alpha)} = \frac{-1}{4\Gamma(3-\alpha)\cos\frac{\pi\alpha}{2}} \begin{cases} 2\left(2^{2-\alpha}\alpha - 4\alpha + 2\right), & \text{for } k = 0 \\ 3^{2-\alpha}\alpha + 2^{2-\alpha}(2-4\alpha) + 7\alpha - 6, & \text{for } k = 1 \\ (k+2)^{2-\alpha}\alpha + (k+1)^{2-\alpha}(2-4\alpha) + k^{2-\alpha}(6\alpha - 6) & \\ \quad + (k-1)^{2-\alpha}(6-4\alpha) + (k-2)^{2-\alpha}(\alpha - 2), & \text{for } k \geq 2 \end{cases} \tag{26}$$

Summarising above calculations we presented difference schemes for the Riesz fractional derivative. It should be noted that expressions (18) and (25) are represented by the weighted sum over discrete values of function u in all node's points. If the index k tends to 0, i.e. in nearest neighbourhood of an arbitrary point x_i, one may observe higher values of $w_k^{(\alpha)}$. Whereas values $w_k^{(\alpha)}$ decrease to zero for next nodes far away from the point x_i.

2.2. Fractional Finite Difference Method

While discretization of the Riesz derivative in space was proposed, in this subsection we describe the finite difference method for equation (1). In comparison to the classical diffusion equation where discretization of the second derivative over space is approximated by the central difference scheme, here we use generalized schemes given by formulae (18) and (25) respectively.

We introduce a temporal grid $0 = t^0 < t^1 < \ldots < t^f < t^{f+1} < \ldots < \infty$ with the grid step $\Delta t = t^{f+1} - t^f$. In a point x_k at the moment of time t^f we denote the function $P(x,t)$ as $P_k^f = P(x_i,t^f)$, for $k \in \mathbf{Z}$ and $f \in \mathbf{N}$.

In the explicit scheme of FDM we replaced (1) by the following formula

$$\frac{P_i^{f+1} - P_i^f}{\Delta t} = K_\alpha \frac{1}{h^\alpha} \sum_{k=-\infty}^{\infty} P_{i+k}^f w_{|k|}^{(\alpha)} \tag{27}$$

and after simplifications we obtained the final form as

$$P_i^{f+1} = \sum_{k=-\infty}^{\infty} P_{i+k}^f s_k^{(\alpha)} \tag{28}$$

where coefficients $s_k^{(\alpha)}$ are following

$$s_k^{(\alpha)} = \begin{cases} 1 + K_\alpha \frac{\Delta t}{h^\alpha} w_0^{(\alpha)}, & \text{for } k = 0 \\ K_\alpha \frac{\Delta t}{h^\alpha} w_{|k|}^{(\alpha)}, & \text{for } k \neq 0 \end{cases} \tag{29}$$

In order to determine stability of the explicit scheme (28) the coefficient $s_0^{(\alpha)}$ defined by (29) must be positive

$$s_0^{(\alpha)} = 1 + K_\alpha \frac{\Delta t}{h^\alpha} w_0^{(\alpha)} > 0 \tag{30}$$

Thus we fixed maximum length of the time step Δt as

$$\Delta t < \frac{-h^\alpha}{K_\alpha w_0^{(\alpha)}} = \begin{cases} \dfrac{2h^\alpha \Gamma(2-\alpha)\cos\frac{\pi\alpha}{2}}{K_\alpha\left(2^{1-\alpha}\alpha - 3\alpha + 2\right)}, & \text{for } 0 < \alpha < 1 \\ \dfrac{2h^\alpha \Gamma(3-\alpha)\cos\frac{\pi\alpha}{2}}{K_\alpha\left(2^{2-\alpha}\alpha - 4\alpha + 2\right)}, & \text{for } 1 < \alpha \le 2 \end{cases} \tag{31}$$

Moreover, the initial condition (10) is introduced directly to every grid nodes at the first time step $t = t^0$. This determines initial values of the function P as

$$P_i^0 = c_0(x_i) \tag{32}$$

This scheme is not easy for application in unbounded domains because it generates infinite number of difference equations. Numerical solution (28) with included the unbounded domain $-\infty < x < \infty$ has no practical implementations in computer simulations. We bounded this domain and at present we solve this problem on the finite domain Ω: $-L \le x \le L$ with absorbing boundaries (9). We choose such fixed value L in order to get enough large domain, in which absorbing boundaries haven't significant influence on solutions. However, this approach introduces certain error. We divide the domain Ω into N sub-domains with the step $h = 2L/N$. Here we have the following modified spatial grid $-L = x_0 < x_1 < \ldots < x_{i-1} < < x_i < x_{i+1} < \ldots < x_N = L$. In comparison to the previous method we assume, in points of the grid placed outside lower and upper limits of the domain Ω and on the boundary nodes x_0 and x_N, that the function P has values 0. Hence, on this background we can modify expression (28) using reduction of sum terms

$$P_i^{f+1} = \begin{cases} \sum_{k=-i}^{N-i} P_{i+k}^f s_k^{(\alpha)}, & \text{for } 0 < i < N \\ 0, & \text{for } i = 0, N \end{cases} \tag{33}$$

In opposite to the classical second derivative which is approximated locally, the Riesz and other fractional derivatives accumulate all values of the function in domain points. Boundary conditions have a direct influence to the numerical solution not only on boundary nodes but also in internal nodes of the domain. Here we use truncation of the function P outside the domain Ω.

3. Simulation results

In this section we present results of calculation obtained by our numerical approach. We try to simulate evolutions of the probability density function over time for $\alpha \in \{0.1, 0.5, 0.99, 1.5, 2\}$ and for $K_\alpha = 1$. We assumed the domain [–25, 25] which we divided into 5000 subintervals ($h = 0.01$). We assumed the absorbing boundary conditions. The initial condition $P_\alpha(x,0^+) = \delta(x)$ is approximated by $P^0_{2500} = 1/h$ and $P_i^0 = 0$, for $i \neq 2500$. Figure 1 shows plots of the probability density function $P_\alpha(x,t)$ over space after $t = 1$ s in the visible interval [–10, 10] (in the logarithmic scale).

It should be noted that for $\alpha = 2$ our solution estimates roughly the Gaussian probability density function and for $\alpha = 1^\pm$ this solution becomes the Cauchy probability density function defined by formulae (8).

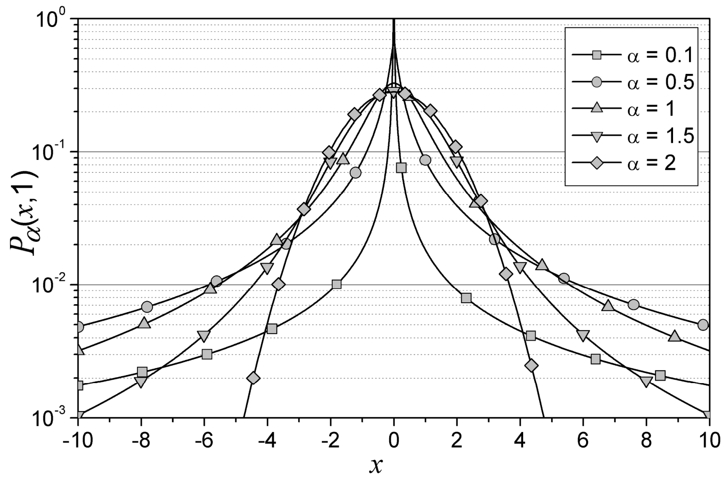

Fig.1. Probability density functions over space for different values of the parameter α and for $t = 1$ s

Conclusions

In summary we proposed the fractional finite difference method for fractional diffusion equations in which the Riesz fractional derivative is included. We analysed the anomalous diffusion equation in linear form in order to compare our numerical results with the analytical solution. We obtained FDM scheme which may generalise classical schemes of FDM. Moreover, for $\alpha = 2$ our solution is the same as the classical finite difference method. We hope that this numerical approach will be successfully applied to fractional partial differential equations having more complex forms, i.e. non-linear forms.

Analysing plots included in this work, we observe that for the case $\alpha < 2$ (the Lévy motion) diffusion is slower than classical diffusion (Brownian motion) at the initial time steps. Nevertheless the probability density function generates a long tail of distribution in the long time limit. This can be associated with rare and extreme events which are characterized by arbitrary very large values of particle jumps. Proposed numerical solution creates a bridge between Gaussian and Cauchy processes.

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