TOROIDAL APPROXIMATION FOR CAPILLARY BRIDGES BETWEEN ELLIPSOIDAL GRAINS

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Abstract. The capillary bridge between ellipsoidal grains is modelled within a toroidal approximation and the cohesive energy as function of grain’s size, wetting angle and liquid content is computed. Some substrate-liquid examples are considered.

Granular materials are common in Nature and they exhibit a variety of unusual phenomena [1]. It is known that many mechanical properties of a granulate change if some liquid is added. Even small amount of interstitial liquid adds an attractive force to the system and then, increase its stability [2-10]. A study of the effects of liquid on static and dynamic properties of dense two-dimensional grain ensembles is of interest in many branches of industry because humidity causes problem in processes such as segregation, transport or packing. The main reason is the internal cohesion due to capillary forces arising from liquid bridges between the grains.

To describe the behaviour of wetting liquid and solid state grain we assume that grains have porous surfaces. Capillary bridges need a certain amount of liquid content to form properly, since the liquid is at first bound on the grain surfaces due to the roughness. Two regimes of the inter-grain adhesive force versus volume of the wetting layer have been considered: the asperity regime for very small amount of liquid and the saturated regime for larger content of liquid. In case of asperity regime the capillary forces come from the fluid accumulated around small number of asperities at which two neighbouring grains in contact. This case has been already discussed in [11]. If all asperities are filled we deal with saturated regime.

Most of grains are not spherical. Thus, it is important to analyze how the macroscopic curvature of contacted grains influences on an inter-grain adhesive energy. Here we consider the ellipsoidal grains. For this shape of grain there are three different grain-to-grain arrangements presented in Figure 1. The geometry of the liquid bridge is characterized by wetting angle, surface tension of liquid, size and shape of grains. Commonly used approach is based on the Laplace-Young equation
\[ \frac{\Delta P}{\gamma} = \frac{d^2 y}{dx^2} \frac{1}{\left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{\frac{3}{2}}} - \frac{1}{\left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{\frac{1}{2}}} \]

where \( \Delta P \) is the capillary suction pressure, \( \gamma \) is the liquid surface tension and unknown function \( y(x) \) is a curvature of the capillary bridge. The Equation (1) cannot be solved analytically [12].

In Figure 1, for the configuration (a) the rotational symmetry of grains and liquid is conserved and so we can approximate the complicated free-liquid surface by simpler surface, i.e. by a toroid. Within the toroidal approximation the Equation (1) becomes

\[ \Delta P = \gamma \left( \frac{1}{r_1} - \frac{1}{r} \right) \]  

(2)

Here \( r_1 \) is the radius of curvature of the liquid bridge at point \( P \) in the horizontal plane and \( r \) is the radius of the curvature in the vertical plane going through the axis of symmetry. An approximate, toroidal capillary bridge is presented in Figure 2.

The geometry of the problem is characterized by wetting angle \( \theta \), liquid volume \( V_k \) and two semi-axis: the minor axis \( \alpha \) and the major axis \( b = k\alpha, k \geq 1 \). Cross-section of the grain and the capillary bridge gives an ellipse and a circle:

\[ \frac{x^2}{a^2} + \frac{(y_1-b)^2}{b^2} = 1 \]

(3)

\[ (x-d)^2 + y_2^2 = r^2 \]

(4)
In order to determine a lateral surface of the liquid bridge it is necessary to estimate the length of an arc between points \( P(x_p, y_p) \) located on two grains. The length of this arc depends on volume \( V_k \) of the capillary bridge. From Equations (3), (4) and Figure 2, we obtain:

\[
x_p = R \sin \beta, \quad y_p = ka - R \cos \beta
\]  
(5)

\[
R = \frac{ka}{\sqrt{k^2 \sin^2 \beta + \cos^2 \beta}}
\]  
(6)

\[
x_p = d - r \sin(\beta + \theta), \quad y_p = r \cos(\beta + \theta)
\]  
(7)

Comparing (5) and (7) we get the relation between the liquid content \( V_k \) (given by \( R, \beta, \theta \)) and the grain size (given by \( \alpha, k \))

\[
r = \frac{ka - R \cos \beta}{\cos(\beta + \theta)}
\]  
(8)

So, using the Guldin-Pappus theorem and the Equation (8) we compute the free surface of the liquid bridge \( S_b \)

\[
S_b = 4\pi a^2 k \frac{I_1 - I_2}{A_1 - A_2} \left( \frac{\pi}{2} \arctan(kc) \right) \left( 1 - \frac{1}{\sqrt{1 + c^2}} \right) \sqrt{1 + (kc)^2}
\]  
(9)
where:

\[
A_1 = \frac{c}{\sqrt{1 + c^2}} - \frac{1}{2} \arctan(c) - \frac{c}{2(1 + c^2)}
\]

\[
A_2 = \frac{k}{2} \left( 1 - \frac{1}{\sqrt{1 + c^2}} \right)^2 \left( 1 + (kc)^2 \right) \left( \frac{\pi}{2} - \arctan(kc) - \frac{kc}{1 + (kc)^2} \right)
\]

\[
I_1 = \frac{c^2}{2(1 + c^2)} + \frac{1}{3(1 + c^2)^{3/2}} - \frac{1}{3}
\]

\[
I_2 = \frac{k}{2} \left( 1 - \frac{1}{\sqrt{1 + c^2}} \right)^2 \left[ \left( k^2 - k^2 - \frac{1}{\sqrt{1 + c^2}} \right) \left( 1 + (kc)^2 \right) \left( \frac{\pi}{2} - \arctan(kc) - \frac{kc}{1 + (kc)^2} \right) - \frac{k}{3} \left( 1 + \frac{1}{\sqrt{1 + c^2}} \right)^2 \right]
\]

and \( c = k \tan(\beta) \).

The cohesive energy \( E \) for the A-A grain arrangement (see Figure 1) is given by

\[
E = \gamma S_b
\]

(10)

where \( \gamma \) is the liquid surface tension and \( S_b \) is given by the Equation (9).

In Table 1 the values of \( S_b \) are given for some \( \theta \)'s grain size \( \alpha \), \( k \) for the liquid amount \( V_k = 10^{-2} \) of the grain’s volume. In Table 2 the same is computed for different substrate and liquid pairs.

**Table 1**

<table>
<thead>
<tr>
<th>( A ) [mm]</th>
<th>0.01</th>
<th>0.02</th>
<th>1</th>
<th>2</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta )</td>
<td>( \pi/40 )</td>
<td>( \pi/40 )</td>
<td>( \pi/12 )</td>
<td>( \pi/11 )</td>
<td>( \pi/11 )</td>
</tr>
<tr>
<td>( S_b ) [mm²]</td>
<td>( 2.5 \cdot 10^{-4} )</td>
<td>( 1.06 \cdot 10^{-3} )</td>
<td>2.43</td>
<td>2.09</td>
<td>8.39</td>
</tr>
</tbody>
</table>

**Table 2**

<table>
<thead>
<tr>
<th>substrate-liquid</th>
<th>( \gamma [10^{-2} \text{ J/m}^2] )</th>
<th>( \theta[^\circ] )</th>
<th>( E = \gamma S_b ) [J]</th>
</tr>
</thead>
<tbody>
<tr>
<td>quartz-water</td>
<td>7.28</td>
<td>0</td>
<td>13.68 \cdot 10^{-8}</td>
</tr>
<tr>
<td>zinc-mercury</td>
<td>48.0</td>
<td>10</td>
<td>90 \cdot 10^{-8}</td>
</tr>
<tr>
<td>polymethacrylate methyl-water</td>
<td>7.28</td>
<td>56</td>
<td>13.90 \cdot 10^{-8}</td>
</tr>
</tbody>
</table>
In conclusion we computed the cohesive energy between ellipsoidal grains due to the small amount of liquid. We approximate free-liquid surface of the bridge by an toroid for (a) grain arrangement, see Figure 1. For (b) and (c) arrangements the toroidal approximations is not valid and other approximation is needed.

References