ON THE CERTAIN APPROACH TO THE HYPERBOLIC HEAT PROPAGATION IN A PERIODICALLY LAMINATED MEDIUM*

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Abstract. The object of our investigations constitutes hyperbolic heat propagation problem in an unbounded periodic laminated medium. We consider solid which has periodically heterogeneous structure in a certain direction and is homogeneous in any perpendicular direction. The main aim of this paper is formulation of a discrete and a discrete-continuum model for the hyperbolic heat transfer problems in the medium under consideration.

1. Introduction

The main attention in this paper is focused on the averaged mathematical modelling of heat transfer problems in periodically laminated media. A fragment of a periodic laminate is presented in Figure 1.

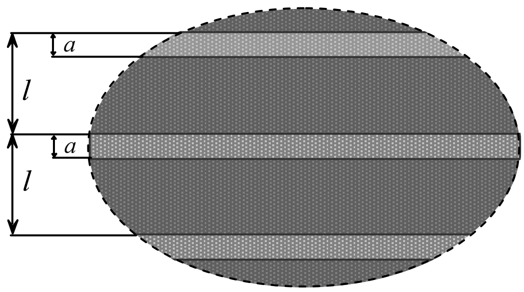

Fig. 1. A fragment of a periodic laminated medium

It is known that the direct approach to heat propagation problems in heterogeneous solids and structures leads to the heat conduction equations with functional, highly oscillating, noncontinuous coefficients. That is why the solution to the initial-boundary-value problems related to this equations constitute complicated computational problems even using computer methods. In order to eliminate this drawback the averaged models (represented by PDEs with constant coefficients) are formulated. The literature related to the averaged modelling of the parabolic heat transfer in laminated structures is very wide. We can mention here models based on

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the homogenisation procedure [1], effective stiffness models [5] or models based on the tolerance averaging technique [8].

The aim of this contribution is to propose a new modelling technique for non-stationary hyperbolic heat transfer processes in the medium under consideration. As the physical basis of our investigations we take Cattaneo constitutive heat conduction equation. We introduce some special description of the periodic laminate geometry and use the concept of periodic simplicial division of the unit cell. In contrast to the alternative models mentioned above, the proposed approach enable us to analyse waves with lengths of an order of the period length.

Notations. Throughout the paper indices a, b, run over $1,\ldots,n$ while A, B and k over $0,1,\ldots,N$ and $1,\ldots,m$, respectively. The summation convention holds for all the indices repeated twice (unless otherwise stated). Symbol t stands for a time coordinate.

2. Basic concepts

We consider a solid which has periodically heterogeneous structure in one direction and is homogeneous in any other perpendicular direction, hence we deal with the plane problem. The geometrical description of a periodic laminated medium is similar to that applied in course of modelling of dynamic problems in composite materials [6]. To make this paper self consistent we recall and modify some of concepts presented in [6].

Let $\{\mathbf{d}_\alpha, \alpha = 1,2\}$ be a vector basis in the physical space E^2 and denote $l_\alpha = |\mathbf{d}_\alpha|$. We introduce the plane Bravais lattice

$$\Lambda = \left\{ \mathbf{z} \in E^2 : \mathbf{z} = \eta_1 \mathbf{d}_1 + \eta_2 \mathbf{d}_2,\ \eta_i = 0, \pm 1, \pm 2, \ldots,\ i = 1,2 \right\}$$

For every subset Ξ and point $\mathbf{p}$ in E^2 and for every $\mathbf{z} \in \Lambda$ we define $\Xi(\mathbf{z}) \equiv \Xi + \mathbf{z}$ and $\mathbf{p}(\mathbf{z}) \equiv \mathbf{p} + \mathbf{z}$, respectively. Let Δ be the parallelogram spanned on the vectors $\mathbf{d}_1$, $\mathbf{d}_2$ with a centre at point $\mathbf{0}$. Let us observe that

1^0 $\bigcup \overline{\Delta(\mathbf{z})} = E^2;\ \ \mathbf{z} \in \Lambda,$

2^0 $\left(\forall \mathbf{z}_1, \mathbf{z}_2 \in \Lambda\right) \left[\mathbf{z}_1 \neq \mathbf{z}_2 \Rightarrow \Delta(\mathbf{z}_1) \cap \Delta(\mathbf{z}_2) = \varnothing\right]$

Thus region Δ will be referred to as a cell in E^2.

By a Δ-periodic simplicial division of E^2 we shall mean the simplicial subdivision of Δ into m simplexes T^k, $k = 1,\ldots,m$, which implies the simplicial division of E^2 into simplexes $T^k(\mathbf{z})$, $\mathbf{z} \in \Lambda$. Let $\mathbf{p}_0^a$, $a = 1,\ldots,n$, be the smallest set of vertices in $\overline{\Delta}$ such that $\left\{\mathbf{p}_0^a(\mathbf{z}) :\ \mathbf{z} \in \Lambda,\ \ a = 1,\ldots,n\right\}$ is a set of all vertices related to the Δ-periodic simplicial division of E^2. We also introduce the system of vectors $\mathbf{d}_A \in \Lambda$,

$A = 1,...,N$, $N > 2$, such that $\mathbf{d}_0 = \mathbf{0}$ and every vertex in $\overline{\Delta}$ can be uniquely represented by the sum $\mathbf{p}_0^a + \mathbf{d}_A$. Setting $I := \{(a, A) \in \{1,...,n\} \times \{0,1,...,N\} : \ \mathbf{p}_0^a + \mathbf{d}_A \in \overline{\Delta}\}$ we shall define $\mathbf{p}_A^a = \mathbf{p}_0^a + \mathbf{d}_A$ for every $(a, A) \in I$. Hence every simplex T^k can be given by $T^k = \mathbf{p}_A^a \mathbf{p}_B^b \mathbf{p}_C^c$, where (a,A), (b,B), (c,C) belong to I.

For an arbitrary function $f(\mathbf{z})$, $\mathbf{z} \in \Lambda$ we define the right- and left hand side finite differences

$$\Delta_A f(\mathbf{z}) = f(\mathbf{z} + \mathbf{d}_A) - f(\mathbf{z}), \ \ \overline{\Delta}_A f(\mathbf{z}) = f(\mathbf{z}) - f(\mathbf{z} - \mathbf{d}_A), \ \ \mathbf{z} \in \Lambda$$

where for $A = 0$ we have $\Delta_0 f(\mathbf{z}) = \overline{\Delta}_0 f(\mathbf{z}) = 0$.

Now, we shall outline some basic physical concepts which were presented in the course of the modelling of hyperbolic heat conduction in the periodic lattice conductor [7]. It is well known that the heat conduction in a solid medium is described by the energy balance law and a certain heat conduction constitutive equation. In our considerations we take into account the Cattaneo constitutive equation with one relaxation time [3]

$$\mathbf{q} + \tau \dot{\mathbf{q}} = -k \nabla \theta$$

In this equation $\mathbf{q}$ stands for the heat flux vector, τ for the relaxation time, k for the modulus of thermal conductivity and θ for the temperature field. Combining this equation with the energy balance law we obtain the hyperbolic heat conduction equation

$$-\nabla(k \nabla \theta) + c\dot{\theta} + \tau c \ddot{\theta} = f + \tau \dot{f}$$

where c stands for the specific heat and f for the heat sources. It can be shown that above equation can be obtained from the principle of stationary action. To this end we introduce the modified temperature field

$$\vartheta := \theta \exp \frac{t}{2\tau} \tag{1}$$

and the lagrangian function of the form

$$L = \frac{1}{8} \frac{c}{\tau} (\vartheta)^2 + \frac{1}{2} \tau c (\dot{\vartheta})^2 - \frac{1}{2} k \nabla \cdot \nabla \vartheta + (f + \tau \dot{f}) \vartheta \exp \frac{t}{2\tau} \tag{2}$$

In this case the variational principle leads to the hyperbolic heat conduction equation

$$-\nabla(k \nabla \vartheta) - \frac{c}{4\tau} \vartheta + \tau c \ddot{\vartheta} = (f + \tau \dot{f}) \exp \frac{t}{2\tau}$$

The above concepts and definitions serve as the basis in formation of approximate mathematical models of heat propagation in laminated media.

3. Discrete model

The first step in the formation of discrete models of an unbounded laminated medium consists a choice of basic vectors $\mathbf{d}_1$, $\mathbf{d}_2$ which generate a basic cell Δ. Subsequently, we introduce a certain Δ-periodic simplicial division of E^2 such that every simplex T^k in the unit cell Δ can be treated, with a sufficient approximation, as homogeneous.
An example of a simplest simplicial discretization of the laminated medium is presented in Figure 2.

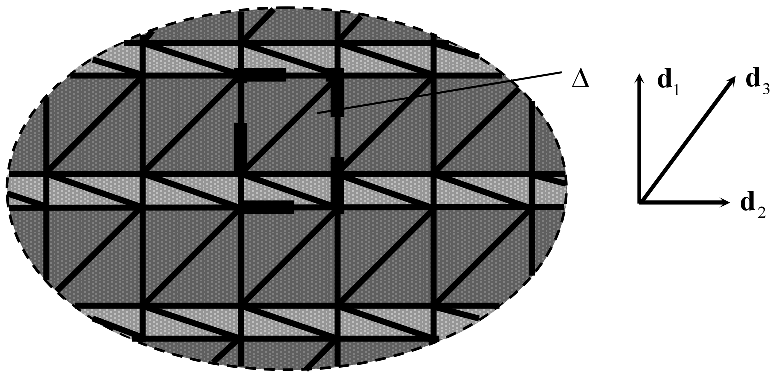

Fig. 2. An example of a first order simplicial discretization

Let us observe that the length of the vector $\mathbf{d}_1$ is determined by the structure of the laminated medium, whilst the length of the vector $\mathbf{d}_2$ is arbitrary. It follows from the fact that the medium is homogeneous in the direction of $\mathbf{d}_2$.

Let $\theta(\mathbf{x},t)$ be a temperature field at time t. We shall denote a temperature at point $\mathbf{p}_A^a(\mathbf{z})$ at time t by $\theta_A^a(\mathbf{z},t) := \theta(\mathbf{p}_A^a(\mathbf{z}),t)$ for every $\mathbf{z} \in \Lambda$, $(a,A) \in I$, and $\theta^a(\mathbf{z},t) := \theta_0^a(\mathbf{z},t)$. The basic assumption of the proposed modelling procedure is that in every problem under consideration the diameters of simplexes T^k have to be taken as sufficiently small when compared to the typical wavelength of the expected temperature pattern. It means that the temperature in the triangle element can be uniquely determined by the temperatures at nodal points

$$\theta(\mathbf{x},t) = \lambda^1(\mathbf{x})\theta_A^a(\mathbf{z},t) + \lambda^2(\mathbf{x})\theta_B^b(\mathbf{z},t) + \lambda^3(\mathbf{x})\theta_C^c(\mathbf{z},t) \tag{3}$$

where $\lambda^1(\mathbf{x})$, $\lambda^2(\mathbf{x})$, $\lambda^3(\mathbf{x})$ are barycentric coordinates of an arbitrary point $\mathbf{x} \in T^k(\mathbf{z})$, $\lambda^1(\mathbf{x}) + \lambda^2(\mathbf{x}) + \lambda^3(\mathbf{x}) = 1$, Zienkiewicz [9]. It follows that the simplexes $T^k(\mathbf{z})$,

$\mathbf{z} \in \Lambda$, are treated as finite elements of E^2. At the same time for every $(a,A) \in I$ we have

$$\theta_A^a(\mathbf{z},t) = \theta^a(\mathbf{z}+\mathbf{d}_A,t) = \theta^a(\mathbf{z},t) + \Delta_A\theta^a(\mathbf{z},t) \tag{4}$$

Similar denotations also hold for the modified temperature ϑ. Let c^k, λ^k, τ stand for the specific heat, coefficient of thermal conductivity, relaxation time of element $T^k(\mathbf{z})$, $\mathbf{z} \in \Lambda$, $k = 1,\ldots,m$, respectively. We also assume the heat sources can depend only on time. Let $|\Delta|$ stand for the area of Δ and $|T^k|$ for the area of triangle T^k.

For every simplex T^k, $k = 1,\ldots,m$ in the unit cell we construct the lagrangian function L_k, which by means of (1)-(4) can be represented in the form

$$\begin{gathered}\mathsf{L}_k = \frac{1}{4\tau}W\left(\vartheta^b, \Delta_A\vartheta^a\right) + \tau W\left(\dot\vartheta^b, \Delta_A\dot\vartheta^a\right) - V\left(\vartheta^a - \vartheta^b, \Delta_A\vartheta^a\right) + \\ + \left(f + \tau\dot f\right)\exp\frac{t}{2\tau}E\left(\vartheta^b, \Delta_A\vartheta^a\right)\end{gathered} \tag{5}$$

where:

$$W\left(\vartheta^b, \Delta_A\vartheta^a\right) = \alpha^k\left(\left(\vartheta^b\right)^2 + \left(\vartheta^b + \vartheta^a + \Delta_A\vartheta^a\right)\left(\vartheta^a + \Delta_A\vartheta^a\right)\right)$$

$$E\left(\vartheta^b, \Delta_A\vartheta^a\right) = \beta^k\left(\vartheta^b + \vartheta^a + \Delta_A\vartheta^a\right)$$

$$V\left(\vartheta^b, \Delta_A\vartheta^a\right) = \gamma^k\left(\vartheta^b - \vartheta^a - \Delta_A\vartheta^a\right)^2$$

where $\alpha^k = \alpha^k\left(c^k, |\Delta|, l^k, |T^k|\right)$, $\beta^k = \beta^k\left(|\Delta|, l^k, |T^k|\right)$, $\gamma^k = \gamma^k\left(|\Delta|, l^k, |T^k|\right)$ are the known coefficients.

Hence we obtain the lagrangian function L of the representative cell Δ

$$\mathsf{L} = \sum_{k=1}^{m}\mathsf{L}_k \tag{6}$$

It can be shown that the variational approach leads to the following system of equations

$$\frac{\partial\mathsf{L}}{\partial\vartheta^a} - \bar\Delta_A\frac{\partial\mathsf{L}}{\partial\Delta_A\vartheta^a} = \frac{d}{dt}\left(\frac{\partial\mathsf{L}}{\partial\dot\vartheta^a} - \bar\Delta_A\frac{\partial\mathsf{L}}{\partial\Delta_A\dot\vartheta^a}\right),\ a = 1,\ldots,n,\ A = 1,\ldots,N \tag{7}$$

Thus, we have arrived at the infinite system of finite difference equations (with respect to argument $\mathbf{z} \in \Lambda$) for the modified temperature $\vartheta^a(\mathbf{z},t)$, $a = 1,\ldots,n$, $\mathbf{z} \in \Lambda$. Equations (7) have to hold for an arbitrary instant t and every $\mathbf{z} \in \Lambda$ and represent the discrete model of the hyperbolic heat conduction of the periodic unbounded laminated medium under consideration.

4. Discrete-continuum models

The main point of this contribution is the formulation of the discrete-continuum model of heat transfer processes in a laminated medium. Because that the medium is discrete in direction of $\mathbf{d}_1$ and continuum in direction of $\mathbf{d}_2$ it is physically reasonable that the functions describing the heat transfer in this structure should have discrete arguments in one direction and continuum in the other. Hence we have the basic modelling assumption in the form

$$\left|\mathbf{d}_1\right| = l = \text{const}, \ \left|\mathbf{d}_2\right| \to 0, \ \left|\mathbf{d}_3\right| \to l$$

This assumption leads to the new lagrangian $\tilde{L}$ obtained from the discrete lagrangian L

$$\tilde{L} = L\left(\vartheta^a, \ \Delta_1\vartheta^a, \partial\vartheta^a, \dot{\vartheta}^a, \ \Delta_1\dot{\vartheta}^a, \partial\dot{\vartheta}^a\right), \ a = 1,...,n$$

Then we obtain the system of equations involving increments of temperatures in the $\mathbf{d}_1$ direction and derivatives of temperatures in the $\mathbf{d}_2$ direction.

$$\frac{\partial\tilde{L}}{\partial\vartheta^a} - \bar{\Delta}_1\frac{\partial\tilde{L}}{\partial\Delta_1\vartheta^a} - \partial\frac{\partial\tilde{L}}{\partial\left(\partial\vartheta^a\right)} = \frac{d}{dt}\left(\frac{\partial\tilde{L}}{\partial\dot{\vartheta}^a} - \bar{\Delta}_1\frac{\partial\tilde{L}}{\partial\Delta_1\dot{\vartheta}^a} - \partial\frac{\partial\tilde{L}}{\partial\left(\partial\dot{\vartheta}^a\right)}\right), \ a = 1,...,n \quad (8)$$

The above equations represent the discrete-continuum model of the hyperbolic heat conduction in the periodically laminated medium.

5. Conclusions

1. A new discrete-continuum model for the analysis of the temperature wave propagation (with one relaxation time) in a laminated rigid conductor is formulated.
2. The model equations are represented by the system of finite-difference equations (in the direction normal to lamina interfaces) combined with partial differential equations.
3. In contrast to the known simplified continuum models (a homogenized model [1] and models based on the tolerance averaging technique, [8]) the proposed model makes it possible to investigate harmonic and exponential temperature waves of the length of an order of lamina thicknesses.

4. Dynamic problems for laminated media can be also investigated using asymptotic methods by an approach detailed in [2] and applied in [4] to the solution of a certain initial-boundary value problem. However, the model proposed in this contribution is more simple and effective then that applied in [4].

Applications of the above theoretical analysis will be exposed in the forthcoming paper.

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