A SEQUENCE OF DISCRETE ALMOST REPRESENTING MEASURES CONVERGENT TO A REPRESENTING MEASURE

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Abstract. We will construct the sequence of discrete measures almost representing points of certain compact convex sets and supported by extreme points of that sets convergent in the weak* topology to a discrete measure which represents point of a compact convex set and is supported by its extreme points.

1. Definitions

By T we will denote the metric space, by X - n-dimensional Euclidean space (although definitions and facts below can be stated in a more general setting).

1) We say that:
   a) a set $A \subset X$ is convex, if whenever it contains two points, it also contains the line segment joining them; „algebraically speaking" A is convex, if $\lambda x + (1-\lambda) y \in A$ whenever $x, y \in A$ and $0 \le \lambda \le 1$;
   b) a point $e \in A$ is an extreme point of A if and only if whenever $e = \lambda x + (1-\lambda) y,\ x, y \in A,\ 0 < \lambda < 1,$ then $x = y = e$ (by ext$A$ we will denote the set of extreme points of A);
   c) the convex hull of $A \subset X$ (denoted by conv$A$) is the set of all convex combinations of points of A

$$cvA := \left\{ x : x = \sum_{i=1}^{n} \lambda_i x_i : x_i \in A,\ \lambda_i \ge 0,\ \sum_{i=1}^{n} \lambda_i = 1 \right\}$$

2) Let $A \subset X$ be a compact convex set, $x \in A$ and $\gamma > 0$. We say that:
   a) a regular probability Borel measure μ on X represents point $x \in A$ if the equality $f(x) = \int_A f d\mu$ holds for all $f \in X^*$;
   b) a regular probability Borel measure μ on X γ- represents point $x \in A$ if the inequality $\left| f(x) - \int_A f d\mu \right| < \gamma$ holds for all $f \in X^*$;

3) Denote by $C_b(X)$ the set of all continuous bounded real functions on X. This set with the supremum norm given by the formula $\|\varphi\| := \sup\{|\varphi(x)| : x \in X\}$ is a Banach space. By $M(X)$ we will denote the space of all probability measures on the σ-algebra $B(X)$ of the Borel subsets of X. Take any such measure and consider the family of sets of the form $V_\mu(\varphi_1, \ldots, \varphi_k, \varepsilon_1, \ldots, \varepsilon_k) :=$ $= \left\{ \nu \in M(X) : \left| \int \varphi_i d\nu - \int \varphi_i d\mu \right| < \varepsilon_i, \quad i = 1, \ldots, k \right\}$ where the functions $\varphi_i \in C_b(X)$, $\varepsilon_i > 0, i = 1, \ldots, k$. The family of all such sets is a base of a topology on $M(X)$, called "the weak*-topology". The generalized sequence (μ_α) of measures converges to the measure μ_0 in this topology iff $\int \varphi d\mu_\alpha \to \int \varphi d\mu_0$ for any $\varphi \in C_b(X)$.
4) A multifunction P is a mapping from the space T into nonempty subsets of a space X. Let $\emptyset \neq A \subset X$. We will use the following notation:

$$P^+(A) := \{x \in X : P(x) \subseteq A\}$$
$$P^-(A) := \{x \in X : P(x) \cap A \neq \emptyset\}$$

We say that multifunction $P : T \to 2^X - \{\emptyset\}$ is:

a) lower semicontinuous, if the set $P^-(V)$ is open in T for every V open in X;

b) upper semicontinuous, if the set $P^+(V)$ is open in T for every V open in X;

c) continuous, if it is both lower- and upper semicontinuous.

5) Let P be a multifunction. A selection of P is a single-valued mapping $p : T \to X$ such that for any $x \in X$ there holds $p(x) \in P(x)$.

2. Facts

In this section we state without proof more or less known facts which will be needed in further considerations.

1) (Krein-Milman theorem) A compact convex set $A \subset X$ is equal to the convex hull of its extreme points (X - finite dimensional).
2) A multifunction $P : T \to 2^X - \{\emptyset\}$ is lower semicontinuous if and only if for every sequence $(t_n) \subset T$ and any point $x_0 \in P(t_0)$ there exists sequence $(x_n) \subset X$ convergent to x_0 and such that $x_n \in P(t_n)$.
3) (Michael selection theorem) Any lower semicontinuous multifunction from a paracompact space into space of nonempty subsets of a Banach space with closed convex values has a continuous selection.

3. Construction

Let T be a metric space, $X - n$ dimensional Euclidean space, $P : T \to 2^X - \{\varnothing\}$ - continuous multifunction with compact convex values. In this case multifunction

$$t \to \operatorname{ext} P(t)$$

is lower semicontinuous (see [3]).
Choose and fix $\gamma > 0$ and a continuous selection $p(\cdot)$ of $P(\cdot)$.

Let (t_n) be a sequence in T, convergent to the point $(t_0) \in T$. Consider point $p(t_0) \in P(t_0)$. By the Krein-Milman theorem there exist points $a_1, ..., a_m \in extP(t_0)$, positive numbers $\lambda_1, ..., \lambda_m, \sum_{i=1}^{m} \lambda_i = 1$, such that $p(t_0) = \sum_{i=1}^{m} \lambda_i a_i$. Then we can check that the discrete measure (i.e. the measure being a convex combination of Dirac measures) $\mu_0 := \sum_{i=1}^{m} \lambda_i \delta_{a_i}$ represents point $p(t_0)$. As multifunction $extP(\cdot)$ is lower semicontinuous, then for any a_i there exists sequence $b_n^i \in \operatorname{ext} P(t_n)$ convergent to a_i. Define measure

$$\mu_n := \sum_{i=1}^{m} \lambda_i \delta_{b_n^i}$$

and let $\varphi \in C_b(X)$. We then have

$$\left| \int \varphi d\mu_n - \int \varphi d\mu_0 \right| \le \sum_{i=1}^{m} \lambda_i \left| \varphi\left(b_n^i\right) - \varphi\left(a_i\right) \right| \xrightarrow[n \to \infty]{} 0$$

This proves that the sequence (μ_n) converges weakly* to the measure (μ_0).
Moreover, for any $f \in X^*$ we have

$$\left| f\left(p(t_n)\right) - \int_{P(t_n)} f d\mu_n \right| \le \left| f\left(p(t_n)\right) - f\left(p(t_0)\right) \right| + \left| f\left(p(t_0)\right) - \int_{P(t_0)} f d\mu_0 \right| +$$

$$+ \left| \int_{P(t_0)} f d\mu_0 - \int_{P(t_n)} f d\mu_n \right|$$

The term on the right converges to 0, the second equals 0 because measure μ_0 represents point $p(t_0)$. For the third term there holds

$$\left| \int_{P(t_0)} f d\mu_0 - \int_{P(t_n)} f d\mu_n \right| \leq \sum_{i=1}^{m} \lambda_i \left| f\left(a_i - b_n^i\right) \right| \xrightarrow[n\to\infty]{} 0$$

Hence there exists natural number n_0 such that for each $n \geq n_0$ the measure μ_n γ-represents point $p(t_n)$. Finally, by construction, $\mu_n\left(extP(t_n)\right) = 1$.

References

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[3] Puchała P., On a lower semicontinuity of a certain multifunction, Scientific Research of the Institute of Mathematics and Computer Science 2003, 1, 2, 153-158.

[4] Webster R., Convexity, Oxford University Press 1994.