A SEQUENCE OF DISCRETE ALMOST REPRESENTING MEASURES CONVERGENT TO A REPRESENTING MEASURE

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Abstract. We will construct the sequence of discrete measures almost representing points of certain compact convex sets and supported by extreme points of that sets convergent in the weak* topology to a discrete measure which represents point of a compact convex set and is supported by its extreme points.

1. Definitions

By $T$ we will denote the metric space, by $X$ - n-dimensional Euclidean space (although definitions and facts below can be stated in a more general setting).

1) We say that:
   a) a set $A \subset X$ is convex, if whenever it contains two points, it also contains the line segment joining them; „algebraically speaking” $A$ is convex, if $\lambda x + (1 - \lambda) y \in A$ whenever $x, y \in A$ and $0 \leq \lambda \leq 1$;
   b) a point $e \in A$ is an extreme point of $A$ if and only if whenever $e = \lambda x + (1 - \lambda) y$, $x, y \in A$, $0 < \lambda < 1$, then $x = y = e$ (by $\text{ext } A$ we will denote the set of extreme points of $A$);
   c) the convex hull of $A \subset X$ (denoted by $\text{conv } A$) is the set of all convex combinations of points of $A$

$$\text{cv } A := \left\{ x : x = \sum_{i=1}^{n} \lambda_i x_i : x_i \in A, \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1 \right\}$$

2) Let $A \subset X$ be a compact convex set, $x \in A$ and $\gamma > 0$. We say that:
   a) a regular probability Borel measure $\mu$ on $X$ represents point $x \in A$ if the equality $f(x) = \int_A f d\mu$ holds for all $f \in X^*$;
   b) a regular probability Borel measure $\mu$ on $X$ - represents point $x \in A$ if the inequality $\left| f(x) - \int_A f d\mu \right| < \gamma$ holds for all $f \in X^*$;
3) Denote by $C_b(X)$ the set of all continuous bounded real functions on $X$. This set with the supremum norm given by the formula $\|\phi\| := \sup \{\phi(x) : x \in X\}$ is a Banach space. By $M(X)$ we will denote the space of all probability measures on the $\sigma$-algebra $B(X)$ of the Borel subsets of $X$. Take any such measure and consider the family of sets of the form $V_\mu(\phi_1, \ldots, \phi_n, \varepsilon_1, \ldots, \varepsilon_k) = \left\{ \nu \in M(X) : \left| \int \phi_i d\nu - \int \phi_i d\mu \right| < \varepsilon_i, \ i = 1, \ldots, k \right\}$ where the functions $\phi_i \in C_b(X)$, $\varepsilon_i > 0$, $i = 1, \ldots, k$. The family of all such sets is a base of a topology on $M(X)$, called “the weak*-topology”. The generalized sequence $(\mu_n)$ of measures converges to the measure $\mu_0$ in this topology iff $\int \phi d\mu_n \to \int \phi d\mu_0$ for any $\phi \in C_b(X)$.

4) A multifunction $P$ is a mapping from the space $T$ into nonempty subsets of a space $X$. Let $\emptyset \neq A \subset X$. We will use the following notation:

$$P^+(A) := \{x \in X : P(x) \subseteq A\}$$
$$P^-(A) := \{x \in X : P(x) \cap A \neq \emptyset\}$$

We say that multifunction $P : T \to 2^X - \{\emptyset\}$ is:

a) lower semicontinuous, if the set $P^-(V)$ is open in $T$ for every $V$ open in $X$;

b) upper semicontinuous, if the set $P^+(V)$ is open in $T$ for every $V$ open in $X$;

c) continuous, if it is both lower- and upper semicontinuous.

5) Let $P$ be a multifunction. A selection of $P$ is a single-valued mapping $p : T \to X$ such that for any $x \in X$ there holds $p(x) \in P(x)$.

2. Facts

In this section we state without proof more or less known facts which will be needed in further considerations.

1) (Krein-Milman theorem) A compact convex set $A \subset X$ is equal to the convex hull of its extreme points ($X$ - finite dimensional).

2) A multifunction $P : T \to 2^X - \{\emptyset\}$ is lower semicontinuous if and only if for every sequence $(t_n) \subset T$ and any point $x_0 \in P(t_0)$ there exists sequence $(x_n) \subset X$ convergent to $x_0$ and such that $x_n \in P(t_n)$.

3) (Michael selection theorem) Any lower semicontinuous multifunction from a paracompact space into space of nonempty subsets of a Banach space with closed convex values has a continuous selection.
3. Construction

Let $T$ be a metric space, $X$- $n$ dimensional Euclidean space, $P:T \to 2^X - \{\emptyset\}$ - continuous multifunction with compact convex values. In this case multifunction

$$t \to \operatorname{ext} P(t)$$

is lower semicontinuous (see [3]).

Choose and fix $\gamma > 0$ and a continuous selection $p(\cdot)$ of $P(\cdot)$.

Let $(t_n)$ be a sequence in $T$, convergent to the point $(t_0) \in T$. Consider point $p(t_n) \in P(t_n)$. By the Krein-Milman theorem there exist points $a_1, \ldots, a_m \in \operatorname{ext} P(t_0)$, positive numbers $\lambda_1, \ldots, \lambda_m$, \( \sum_{i=1}^{m} \lambda_i = 1 \), such that $p(t_0) = \sum_{i=1}^{m} \lambda_i a_i$. Then we can check that the discrete measure (i.e. the measure being a convex combination of Dirac measures) $\mu_0 := \sum_{i=1}^{m} \lambda_i \delta_{a_i}$ represents point $p(t_0)$. As multifunction $\operatorname{ext} P(\cdot)$ is lower semicontinuous, then for any $a_i$ there exists sequence $b_{i_n} \in \operatorname{ext} P(t_n)$ convergent to $a_i$. Define measure

$$\mu_n := \sum_{i=1}^{m} \lambda_i \delta_{b_{i_n}}$$

and let $\varphi \in C_b(X)$. We then have

$$\left| \int \varphi \, d\mu_n - \int \varphi \, d\mu_0 \right| \leq \sum_{i=1}^{m} \lambda_i \left| \varphi(b_{i_n}) - \varphi(a_i) \right| \xrightarrow{n \to \infty} 0$$

This proves that the sequence $(\mu_n)$ converges weakly* to the measure $(\mu_0)$.

Moreover, for any $f \in X^*$ we have

$$\left| f(p(t_n)) - \int f \, d\mu_n \right| \leq \left| f(p(t_n)) - f(p(t_0)) \right| + \left| f(p(t_0)) - \int f \, d\mu_0 \right| + \left| \int f \, d\mu_0 - \int f \, d\mu_n \right|$$

The term on the right converges to 0, the second equals 0 because measure $\mu_0$ represents point $p(t_0)$. For the third term there holds
\[
\left| \int_{\rho V(0)} fd\mu_0 - \int_{\rho V(n)} fd\mu_n \right| \leq \sum_{i=1}^m \lambda_i \left| f(a_i - b_i) \right| \xrightarrow{n \to \infty} 0
\]

Hence there exists natural number \( n_0 \) such that for each \( n \geq n_0 \) the measure \( \mu_n \gamma \)-represents point \( p(t_n) \). Finally, by construction, \( \mu_n(\text{extP}(t_n)) = 1 \).

References