# THE ANALYSIS OF SOME MODELS FOR CLAIM PROCESSING IN INSURANCE COMPANIES

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**Abstract.** In the present paper the analysis of models for claim processing in insurance companies when the total number of insurance contracts may be a function of time is carried out. Closed by the structure queueing networks with bounded time of claims stay in the queues of processing systems serves as models for claim processing.

## 1. Introduction

The analysis of some mathematical models for eqitype and multi-type claims processing was carried out in the paper [1] already. Let an insurance company consists of a central department and $n-1$ sister companies. Every advanced claim passes two stages of processing - the estimation stage in any of sister companies and the stage of payment in the central department. Assume that the total time for waiting of an insurer who advance a claim in the queue fo *i*th sister company and the time, he needs for application in the other sister company is distributed according to the exponential rule with parameter $\nu_i$, $i=\overline{1,n-1}$. The insurer who is not served in the *i*th sister company, advances a claim in the *j*th sister company with probability $q_{ij}$, $i,j=\overline{1,n-1}$. The queueing network with bounded time for claim waiting in the queues of processing systems serves as the probabilistic model for claim processing in this case. Let us describe such a network.

Consider a closed queueing network, which consists of $n+1$ processing systems $S_0,S_1,...,S_n$, in which $K$ eqitype claims circulate. The system $S_i$ consists of $m_i$ identical processing lines, the time of processing in each line is distributed according to the exponential rule with average $\mu_i^{-1}$, $i=\overline{0,n}$. Besides suppose, that the time of stay of claims in the queue of *i*th processing system is a variate which is distributed according to the exponential rule with a parameter $\nu_i$, $i=\overline{0,n}$. The claims for processing are chosen according to the FIFO discipline. The claim, processing of which in the system $S_i$ is finished, passes to the queue of the system $S_j, i,j=\overline{0,n}$ with probability $p_{ij}$, and the claim, the waiting time of which is elapsed, passes to the queue of the system $S_j$ with probability $q_{ij}$, $i,j=\overline{0,n}$. In the general case the matrices of transitions $P = \|p_{ij}\|$, $Q = \|q_{ij}\|$, $i, j = \overline{0,n}$, are not identical and they are the matrices of transition probabilities of irreducible Markov chain.

The vector $k(t) = (k_0(t), k_1(t), ..., k_n(t))$, where $k_i(t)$ is a number of claims in the system $S_i$ at the moment of time $t$, $i = \overline{0,n}$, forms $(n + 1)$-dimensional Markov process with continuous time and finite number of states. Obviously, $k_0(t) = K - \sum_{i=1}^{n} k_i(t)$, where $K$ is a number of claims in the system, since the system is closed.

In [2] it is determined, that the density of probability distribution of the relative variables vector $\xi(t) = \left(\frac{k_0(t)}{K}, \frac{k_1(t)}{K}, ..., \frac{k_n(t)}{K}\right)$ satisfies the Kolmogorov-Fokker--Planc equation to $O(\varepsilon^2)$, where $\varepsilon = \frac{1}{K}$

$$\frac{\partial p(x,t)}{\partial t} = -\sum_{i=0}^{n} \frac{\partial}{\partial x_i}\left(A_i(x)p(x,t)\right) + \frac{\varepsilon}{2}\sum_{i,j=0}^{n} \frac{\partial^2}{\partial x_i \partial x_j}\left(B_{ij}(x)p(x,t)\right) \tag{1}$$

Where:

$$A_i(x) = \sum_{j=0}^{n} [\mu_j p_{ji}^* \min(l_j, x_j) + (x_j - l_j)\nu_j q_{ji}^* u(x_j - l_j)] \tag{2}$$

$$B_{ii}(x) = \sum_{j=0}^{n} [\mu_j r_{ji} \min(l_j, x_j) + (x_j - l_j)\nu_j r_{ji}^* u(x_j - l_j)]$$

$$B_{ij}(x) = -2\mu_i p_{ij} \min(l_i, x_i) - 2(x_i - l_i)\nu_i q_{ij} u(x_i - l_i)$$

$$p_{ji}^* = r_{ji} = p_{ji},\ q_{ji}^* = r_{ji}^* = q_{ji},\ i \neq j$$

$$p_{ji}^* = -r_{ji} = -1 + p_{ii},\ q_{ji}^* = -r_{ji}^* = -1 + q_{ii},\ i = j$$

$u(x)$ is a Heavyside function.

As it was shown in [2] from the equation (2) it follows that to the same accuracy the components of the vector $n(t) = \left(n_0(t), n_1(t), ..., n_n(t)\right)$, where $n_i(t) = M\left\{\frac{k_i(t)}{K}\right\}$, $n_i(t) = M\left\{\frac{k_i(t)}{K}\right\}$ $i = \overline{0,n}$, can be determined from the differential equations set

$$\frac{dn_i(t)}{dt} = A_i(n(t)) = \sum_{j=0}^{n} [\mu_j p_{ji}^* \min(l_j, n_j(t)) + (n_j(t) - l_j)\nu_j q_{ji}^* u(n_j(t) - l_j)] \quad (3)$$

$$i = \overline{0,n}$$

The equations set (3) can be obtained from the equation (1) if one performs the average-out operation on its left and right parts, i.e. one should integrate its both parts in the range from 0 to 1 by each component $x_i$, $i = \overline{0,n}$, and multiply the integrable function on a corresponding component in addition. Then the integral from $O(\varepsilon^2)$ gives us the expression of the order $O(\varepsilon^2)$. The right parts of the equations (3) are piecewise discontinuous functions. Using the decomposition of the phase space one can determine an explicit form of the set (3) in the domains of continuity of its right part

$$\frac{dn_i(t)}{dt} = \sum\nolimits_0 [\mu_j p_{ji}^* l_j + (n_j(t) - l_j)\nu_j q_{ji}^*] + \sum\nolimits_1 \mu_j p_{ji}^* n_j(t), \quad i = \overline{0,n}$$

where

$$\sum\nolimits_0 = \sum_{j \in \Omega_0(t)}, \quad \sum\nolimits_1 = \sum_{j \in \Omega_1(t)}, \quad \Omega_0(t) = \{j : l_j < n_j(t) \le 1\}, \quad \Omega_1(t) = \{j : 0 \le n_j(t) \le l_j\}$$

are non-overlapping sets of the indices of the vector $n(t)$ components.

The above described queueing network may be used as a generalized model of the claims processing in an insurance company, described in [1]. Let an insurance company concluded $K$ equitype insurance contracts with insurers. Let $m_i$ company employees (estimators) are occupied with claims estimation and $m_n$ company employees are occupied with claims payment. Assume that a probability of claim advancing in the $i$th sister company on the interval of time $[t, t + \Delta t]$ equals to $\mu_0(t) p_{0i} \Delta t + o(\Delta t)$, where $\mu_0(t)$ is a piecewise constant function with two intervals of constancy, which characterize the intensity of claims entry:

$$\mu_0(t) = \begin{cases} \mu_{01}, & t \in [0, T/2] \\ \mu_{02}, & t \in (T/2, T] \end{cases}$$

Claims processing times by the estimators in the $i$th sister company and claims processing times by the estimators in the central department are distributed according to the exponential rule with intensities $\mu_i$, $i = \overline{1, n-1}$, and $\mu_n$ correspondingly. Besides, the total time of the insurer stay, who advances a claim, in the queue of the $i$th sister company and the time he needs for application in the other sister company are also distributed according to the exponential rule with other parameter $v_i$, i.e. an insurer, who is not served in the *i*th sister company with probability $q_{ij}$ advances a claim in the *j*th sister company, $i,j=\overline{1,n-1}$.

The company state at the moment of time $t$ may be described by the vector $k(t)=(k_1(t),k_2(t),...,k_n(t))$, where $k_i(t)$ and $k_n(t)$ are the number of claims which are in the *i*th sister company, $i=\overline{1,n-1}$, and in the central department correspondingly. The company performance (average inputs of the company on the intervals of time $[0,T/2]$, $(T/2,T]$ correspondingly) may be described by the functional [1; 2]

$$W(T)=W(T,m_1,...,m_n)=\frac{1}{T}\int_0^T\left[K\sum_{i=1}^{n}(d_i n_i(t)+E_i l_i)\right]dt \tag{4}$$

where: $d_i$, $\mathring{A}_i$, $i=\overline{1,n}$ - cost coefficients. We are interested in the problem of determination of estimators' number on the intervals of time $[0,T/2]$ and $(T/2,T]$, which minimizes the average inputs (4) under restrictions on the average claims number $Kn_i(t)$, which are on the different processing stages.

Naturally, the closed queueing network with bounded time of claims stay in the queues, which consists of the central processing system $S_n$ (central department), $n-1$ outlying processing systems $S_1,S_2,...,S_{n-1}$ (sister companies) and the system $S_0$, which corresponds to the external environment (source of claims entry) may serve as a probabilistic model of claims processing, $m_0 = K$. Transitions probabilities between systems are as follows: $p_{0i}\neq 0$, $p_{in}=1$, $i=\overline{1,n-1}$, $\sum_{i=1}^{n-1}p_{0i}=1$, $p_{ij}=0$ in other cases; $q_{ij}\neq 0$, $i\neq j$, $\sum_{j=1}^{n-1}q_{ij}=1$, $i,j=\overline{1,n-1}$, $q_{ij}=0$ in other cases.

## 2. Analysis of the generalized model

The present model may be generalized on the case of the multy-type claims, when their total number does not depend on time. Let the total number of insurance contracts concluded to the moment of time $t$, $t\in[0,T]$ be defined by a function $\sum_{c=1}^{r-1}K_c(t)=K(t)$, where $K_c(t)$ is a number of contracts of the type $c$, $c=\overline{1,r-1}$. Suppose that an insurance company consists of $n$ sister companies, which generally speaking may differ in sets of claim types, which they can serve, as well as in number of employees. Assume that the probability of the type $c$ claim advanced in the number $i$ sister company on the interval of time $[t, t+\Delta t]$ is $\mu_{0ic}(t)\Delta t + o(\Delta t) = \mu_{0c}(t) p_{0ic} \Delta t + o(\Delta t)$, where $\mu_{0c}(t)$ is an intensity of filing of the type $c$ claim, $\sum_{i=1}^{n} p_{0ic} = 1$, $c = \overline{1, r-1}$, $i = \overline{1, n}$. Every claim advanced in the $i$th sister company may be in two operating steps: the stage of estimation and the stage of payment. Let $m_{ic}$ company's specialists (estimators) of the $i$th sister company be occupied with estimation of the type $c$ claims, and let the time of claim processing be distributed according to the exponential rule with the average value $\mu_{ic}^{-1}$, $i = \overline{1, n}$, $c = \overline{1, r-1}$. The claim which passed the estimation stage in the $i$th sister company comes in to the payment department of the same sister company, where it is processed by one of $m_{ir}$ cashiers and the time required to the claim payment by every cashier is distributed according to the exponential rule with the average value $\mu_{ic}^{-1}$, $i = \overline{1, n}$ as well. Besides assume that the time of waiting of an insurer who advance the type $c$ claim in the $i$th sister company and the time, he needs for application in the other sister company is bounded by a variate which is distributed according to the exponential rule with parameter $\nu_{ic}$, $i = \overline{1, n}$, $c = \overline{1, r-1}$. That is, the insurer who is not served in the $i$th sister company, advances the type $c$ claim in the sister company number $j$ with probability $q_{icjc}$ and this sister company estimates the claims of the such type, $i, j = \overline{1, n}$, $c = \overline{1, r-1}$.

The state of the insurance company at the moment of time $t$ may be described by the vector

$$k(t) = \left(k_{11}(t), k_{12}(t), \ldots, k_{1r-1}(t), k_{1r}(t), \ . \ . \ . \ , k_{n1}(t), k_{n2}(t), \ldots, k_{nr-1}(t), k_{nr}(t)\right)$$

where $k_{ic}(t)$ is a number of type $c$ claims, which are in the estimation stage in the $i$th sister company at the moment of time $t$, $i = \overline{1, n}$, $c = \overline{1, r-1}$; $k_{ir}(t)$ is a number claims which are in the payment stage in the $i$th sister company at the moment of time $t$, $i = \overline{1, n}$, $k_0(t) = K(t) - \sum_{i=1}^{n} \sum_{c=1}^{r} k_{ic}(t)$ is a number of contracts which do not need advancing of the claim at the moment of time $t$ (insured accident did not occur).

The company's average loss from one insurer on the interval of time $[T_1, T_2]$ may be defined by a functional [1]

$$W(T_1, T_2, m_{11}, \ldots, m_{nr}) = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \left[ \sum_{i=1}^{n} \sum_{c=1}^{r} \left( d_{ic} n_{ic}(t) + E_{ic} l_{ic}(t) \right) \right] dt \qquad (5)$$

where: $n_{ic}(t)=M\left(\frac{k_{ic}(t)}{K(t)}\right)$, $l_{ic}(t)=\frac{m_{ic}}{K(t)}$, coefficients $d_{ic}$, $E_{ic}$ have cost meaning, $i=\overline{1,n}$, $c=\overline{1,r}$. We are interested in the problem of determination on the interval of time $[T_1,T_2]$ of the estimators and cashiers number, which minimize the average loss (5) under restriction on the average number of claims $K(t)n_{ic}(t)$, which are in the various operating steps $i=\overline{1,n}$, $c=\overline{1,r}$. Usually the queues of insurers occurs as a rule in the estimation stages, so we will solve the following problem:

$$\begin{cases} W(T_1,T_2,m_{11},...,m_{nr}) \longrightarrow \min\limits_{m_{ic},i=\overline{1,n},c=\overline{1,r}} \\ \frac{1}{T_2-T_1}\int_{T_1}^{T_2} n_{ic}(t)dt > \frac{1}{T_2-T_1}\int_{T_1}^{T_2} l_{ic}(t)dt,\ i=\overline{1,n},\ c=\overline{1,r-1} \\ \frac{1}{T_2-T_1}\int_{T_1}^{T_2} n_{ir}(t)dt \le \frac{1}{T_2-T_1}\int_{T_1}^{T_2} l_{ir}(t)dt,\ i=\overline{1,n} \end{cases} \tag{6}$$

A closed by the structure queueing network may serve as a model of the described process, and the total number of different-type claims in it is described by the function of time *K*(*t*). The network consists of $nr+1$ systems $S_0,S_{11},S_{12},...,S_{1r}$, $...,S_{n1},S_{n2},...,S_{nr}$, the system $S_0$ corresponds to the external environment (the claim is not advanced) and the time of claim waiting in the systems' $S_{ic}$, $i=\overline{1,n}$, $c=\overline{1,r-1}$ queues is bounded by an exponential variate. The transition probabilities between the network's systems are $p_{0ic}\neq 0$, $p_{icir}=p_{ir0}=1$, $i=\overline{1,n}$, $c=\overline{1,r-1}$. Besides the following claim transitions from the systems' queues are possible: $q_{icjc}\neq 0$, $i,j=\overline{1,n}$, $c=\overline{1,r-1}$, $q_{icjs}=0$ in the other cases. Service disciplines in the network's systems are FIFO. The other parameters are described before.

Using the method described in [2], it is determined that the density of probabilities distribution of the vector relative variables $\xi(t)=\left(\frac{k(t)}{K(t)}\right)$ satisfies to $O(\varepsilon^2(t))$, where $\varepsilon(t)=\frac{1}{K(t)}$, to the differential equation in the partial derivatives of the second order:

$$\frac{\partial p(x,t)}{\partial t} = -\sum_{i=1}^{n}\sum_{c=1}^{r}\frac{\partial}{\partial x_{ic}}\left(A_{ic}(x,t)p(x,t)\right) + \frac{\varepsilon(t)}{2}\sum_{i,j=1}^{n}\sum_{c,s=1}^{r}\frac{\partial^2}{\partial x_{ic}\partial x_{js}}\left(B_{icjs}(x,t)p(x,t)\right) + nr\frac{K'(t)}{K(t)}p(x,t) \tag{7}$$

where:

$$A_{ic}(x,t) = \sum_{j=1}^{n}\sum_{s=1}^{r}[\nu_{js}q^{*}_{jsic}(x_{js}-l_{js})u(x_{js}-l_{js}) + \mu_{js}p^{*}_{jsic}\min(x_{js},l_{js})] + \mu_{0ic}\left(1-\sum_{j=1}^{n}\sum_{s=1}^{r}x_{js}\right) \tag{8}$$

$$B_{icic}(x,t) = \sum_{j=1}^{n}\sum_{s=1}^{r}[\nu_{js}q^{**}_{jsic}(x_{js}(t)-l_{js}(t))u(x_{js}(t)-l_{js}(t)) + \mu_{js}p^{**}_{jsic}\min(x_{js}(t),l_{js}(t))] + \mu_{0ic}\left(1-\sum_{j=1}^{n}\sum_{s=1}^{r}x_{js}(t)\right)$$

$$B_{icjs}(x,t) = -2\nu_{ic}q_{icjs}(x_{ic}(t)-l_{ic}(t))u(x_{ic}(t)-l_{ic}(t)) - 2\mu_{ic}p_{icjs}\min(x_{ic}(t),l_{ic}(t))$$

$$\mu_{0ir}(t) = 0,\ l_{ic}(t) = \frac{m_{ic}}{K(t)},\ i=\overline{1,n},\ c=\overline{1,r}$$

$$q^{*}_{jsic} = q^{**}_{jsic} = \begin{cases} q_{jsic},\ s=c,\ i\neq j,\ i,j=\overline{1,n},\ c=\overline{1,r-1} \\ 0,\ \text{in other cases}\end{cases}$$

$$p^{*}_{jsic} = p^{**}_{jsic} = \begin{cases} 1,\ i=j,\ c=r,\ i,j=\overline{1,n},\ s=\overline{1,r-1} \\ 0,\ \text{in other cases}\end{cases}$$

$$q^{*}_{jsic} = -q^{**}_{jsic} = -1,\ i=j,\ s=c,\ i=\overline{1,n},\ c=\overline{1,r-1}$$

$$p^{*}_{jsic} = -p^{**}_{jsic} = -1,\ i=j,\ s=c,\ i=\overline{1,n},\ c=\overline{1,r}$$

The equation (7) when $K(t) = K =$ const coincide with the well-known Kolmogorov-Fokker-Planc equation for the density of probabilities of $nr$-dimensional Markov process. Using the Gaussian approximation method for the equation (7) one can obtain the usual differential equations set for the components of the vector $n(t)$ and the solution of the problem (6).

## 3. Example

Consider the case when an insurance company, which consists of two sister companies sets up a equitype contracts. For the solution of the problem (6) it is necessary to find components of the vector $n(t)=\left(n_{11}(t),n_{12}(t),n_{21}(t),n_{22}(t)\right)$. They satisfy the equations set, which follows from (8)

$$\begin{cases} n'_{11}(t)=(-\nu_{11}-\mu_{011}(t))n_{11}(t)-\mu_{011}(t)n_{12}(t)+(\nu_{21}-\mu_{011}(t))n_{21}(t)-\mu_{011}(t)n_{22}(t)+ \\ \qquad +l_{11}\nu_{11}-\mu_{11}l_{11}-l_{21}\nu_{21}+\mu_{011}(t)-\varepsilon(t)(\nu_{11}-\nu_{21}-\mu_{011}) \\ n'_{12}(t)=-\mu_{12}n_{12}(t)+\mu_{11}l_{11}-\varepsilon(t)\mu_{12} \\ n'_{21}(t)=(\nu_{11}-\mu_{021}(t))n_{11}(t)-\mu_{021}(t)n_{12}(t)+(-\nu_{2}-\mu_{021}(t))n_{21}(t)-\mu_{021}(t)n_{22}(t)+ \\ \qquad +l_{21}\nu_{21}-l_{11}\nu_{11}-\mu_{21}l_{21}+\mu_{021}(t)-\varepsilon(t)(\nu_{21}-\nu_{11}-\mu_{021}(t)) \\ n'_{22}(t)=-\mu_{22}n_{22}(t)+\mu_{21}l_{21}-\varepsilon(t)\mu_{22} \end{cases}$$

In the case when the intensity $\mu_{0ic}(t)$ is piecewise constant and $K(t)=\dfrac{a}{e\sin(bt)+d}$, $a,e,b,d$ are constants on every interval of intensity constancy of incoming flow under defined initial conditions we obtain that all $n_{ic}(t)$ have the following type $n_{ic}(t)=\sum_{p,z=1}^{2}[m_{zp}\alpha_{iczp}\sin(bt)+m_{zp}\beta_{iczp}\cos(bt)+\sum_{k=1}^{5}m_{zp}\gamma_{iczpk}e^{\lambda_k t}]$, $\lambda_5=0$, $i,c=1,2$.

The functional $W(T_1,T_2,m_{11},m_{12},m_{21},m_{22})=\sum_{i,c=1}^{2}g_{ic}m_{ic}+g_0$ is a linear function of $m_{ic}$, $i,c=1,2$. Restrictions of the optimization problems are linear as well: $\sum_{i,c=1}^{2}h_{ick}m_{ic}+h\le 0$, $k=\overline{1,4}$. So in the considered case the problem (6) is the linear programming problem.

**Example.** Let a n insurance company, which consists of two sister companies, sets up the equitype insurance contracts, and let the total number of contracts be described by the function of time $K(t)=\dfrac{100000}{3\sin(2\pi t/364)+5}$, $t\in[0,364]$, and its functioning is described by the following parameters:

$$\mu_{01}(t)=\begin{cases}\mu^{*}_{01}, & t\in[0,182] \\ \mu^{**}_{01}, & t\in(182,364]\end{cases}$$

$$\mu_{01}^* = 0.003,\ \mu_{02}^* = 0.005,\ \mu_{11} = 15,\ \mu_{21} = 25,\ \mu_{12} = 70,\ \mu_{22} = 80,$$
$$E_{11} = 20,\ E_{21} = 20,\ E_{12} = 10,\ E_{22} = 10,\ d_{11} = 5,\ d_{21} = 5,\ d_{12} = 2,\ d_{22} = 2,$$
$$p_{011} = 0.4,\ \nu_{11} = 0.3,\ \nu_{21} = 0.4$$

Solving the problem (6) on every interval of intensity constancy $\mu_{01}(t)$ we obtain that 2 estimators and 1 cashier should operate in the first sister company on the interval of time $[0,182]$, and 2 estimators and 1 cashier - in the second sister company. 5 estimators and 1 cashier should operate in the first sister company on the interval of time (182,364] and 4 estimators and 1 cashier - in the second sister company.

## References

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