MODELLING OF TEMPERATURE DISTRIBUTION IN HEATING TISSUE WITH REGARD TO THE SHAPE SENSITIVITY ANALYSIS

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Abstract. In the paper the numerical analysis of heat transfer process proceeding in domain of biological tissue is presented. In particular, the 1D problem is considered. The temperature distribution results from the action of the disk emanating the constant heat flux. The problem corresponds, among others, to chronic heating caused by implants being a heat source [1]. The sensitivity analysis of this process with respect to the thickness of biological tissue is presented. On the stage of numerical computations the boundary element method is used. In the final part of the paper the results obtained are shown.

1. Formulation of the problem

We consider the following bioheat transfer equation [2]

$$L_0 < x < L_1 : \quad \lambda \frac{\mathrm{d}^2 T(x)}{\mathrm{d} x^2} + k\left[T_B - T(x)\right] + Q_m = 0 \tag{1}$$

where: λ, W/mK, is the thermal conductivity, $k = c_B G_B$ (c_B, J/m^{3}K, is the volumetric specific heat of blood, G_B, m^3 blood/s/m^3 tissue, is the perfusion rate), T_B is the blood temperature, Q_m, W/m^3, is the metabolic heat source, T is the temperature, x is the geometrical co-ordinate, while $L_1 - L_0$, m, is the distance between superficial heat source and the arbitrary assumed internal boundary of tissue.

The equation (1) is supplemented by boundary conditions

$$x = L_0 : \quad q(L_0) = -\lambda \left.\frac{\mathrm{d} T}{\mathrm{d} x}\right|_{x = L_0} = q_b \tag{2}$$

where q_b is the given heat flux, and

$$x = L_1 : \quad T\left(L_1\right) = T_b \tag{3}$$

where T_b is the known boundary temperature.

The aim of this paper is to estimate the change of temperature due to a change of the biological tissue thickness.

We assume that $b = L_0$ is the shape design parameter. Using the concept of material derivative we can write [3, 4]

$$\frac{\mathbf{D}T}{\mathbf{D}b} = \frac{\partial T}{\partial b} + \frac{\partial T}{\partial x}v \tag{4}$$

where $v = v(x, b)$ is the velocity associated with design parameter b.
Because (c.f. equation (4))

$$\frac{\mathbf{D}}{\mathbf{D}b}\left(\frac{\partial T}{\partial x}\right) = \frac{\partial}{\partial b}\left(\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right)v = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial b}\right) + \frac{\partial^2 T}{\partial x^2}v \tag{5}$$

and

$$\frac{\partial}{\partial x}\left(\frac{\mathbf{D}T}{\mathbf{D}b}\right) = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial b} + \frac{\partial T}{\partial x}v\right) = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial b}\right) + \frac{\partial^2 T}{\partial x^2}v + \frac{\partial T}{\partial x}\frac{\partial v}{\partial x} \tag{6}$$

therefore

$$\frac{\mathbf{D}}{\mathbf{D}b}\left(\frac{\partial T}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\mathbf{D}T}{\mathbf{D}b}\right) - \frac{\partial T}{\partial x}\frac{\partial v}{\partial x} \tag{7}$$

Using formula (7) one has

$$\frac{\mathbf{D}}{\mathbf{D}b}\left(\frac{\partial^2 T}{\partial x^2}\right) = \frac{\mathbf{D}}{\mathbf{D}b}\left[\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right)\right] = \frac{\partial}{\partial x}\left[\frac{\mathbf{D}}{\mathbf{D}b}\left(\frac{\partial T}{\partial x}\right)\right] - \frac{\partial^2 T}{\partial x^2}\frac{\partial v}{\partial x} \tag{8}$$

and next

$$\begin{gathered}\frac{\mathbf{D}}{\mathbf{D}b}\left(\frac{\partial^2 T}{\partial x^2}\right) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(\frac{\mathbf{D}T}{\mathbf{D}b}\right) - \frac{\partial T}{\partial x}\frac{\partial v}{\partial x}\right] - \frac{\partial^2 T}{\partial x^2}\frac{\partial v}{\partial x} = \\ \frac{\partial^2}{\partial x^2}\left(\frac{\mathbf{D}T}{\mathbf{D}b}\right) - 2\frac{\partial^2 T}{\partial x^2}\frac{\partial v}{\partial x} - \frac{\partial T}{\partial x}\frac{\partial^2 v}{\partial x^2}\end{gathered} \tag{9}$$

Presented above formulas are necessary in order to realize the shape sensitivity analysis of bioheat transfer process.

2. Shape sensitivity analysis

If the direct approach of sensitivity method is applied [3, 4] then the equation (1) is differentiated with respect to shape parameter b, and then

$$L_0 < x < L_1 : \quad \lambda\frac{\mathbf{D}}{\mathbf{D}b}\left(\frac{\mathrm{d}^2 T}{\mathrm{d}x^2}\right) - k\frac{\mathbf{D}T}{\mathbf{D}b} = 0 \tag{10}$$

or using the formula (9)

$$\lambda \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\mathbf{D}T}{\mathbf{D}b}\right) - 2\lambda \frac{\mathrm{d}^2 T}{\mathrm{d}x^2}\frac{\partial v}{\partial x} - \lambda \frac{\mathrm{d}T}{\mathrm{d}x}\frac{\partial^2 v}{\partial x^2} - k\frac{\mathbf{D}T}{\mathbf{D}b} = 0 \tag{11}$$

From equation (1) results that

$$\lambda \frac{\mathrm{d}^2 T}{\mathrm{d}x^2} = -k\left(T_B - T\right) - Q_m \tag{12}$$

so the equation (11) can be written in the form

$$\lambda \frac{\mathrm{d}^2 U(x)}{\mathrm{d}x^2} - kU(x) + 2\left[kT_B - kT(x) + Q_m\right]\frac{\partial v}{\partial x} - \lambda \frac{\mathrm{d}T(x)}{\mathrm{d}x}\frac{\partial^2 v}{\partial x^2} = 0 \tag{13}$$

where $U = \mathbf{D}T/\mathbf{D}b$ is the sensitivity function.
The boundary condition should be also differentiated. Because (c.f. formula (5))

$$\frac{\mathbf{D}}{\mathbf{D}b}\left(\lambda\frac{\mathrm{d}T}{\mathrm{d}x}\right) = \lambda\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathbf{D}T}{\mathbf{D}b}\right) - \frac{\mathrm{d}T}{\mathrm{d}x}\frac{\partial v}{\partial x}\right] = \lambda\frac{\mathrm{d}U}{\mathrm{d}x} - \lambda\frac{\mathrm{d}T}{\mathrm{d}x}\frac{\partial v}{\partial x} \tag{14}$$

so the boundary condition (2) takes a form

$$x = L_0 : \quad W(L_0) = q_0 \left.\frac{\partial v}{\partial x}\right|_{x = L_0} \tag{15}$$

where $W(x) = -\lambda \mathrm{d}U(x)/\mathrm{d}x$, while

$$x = L_1 : \quad U(L_1) = 0 \tag{16}$$

We assume that

$$v = v(x, b) = \frac{L_1 - x}{L_1 - b} \tag{17}$$

and then the additional problem connected with the shape sensitivity analysis with respect to the parameter b takes a form

$$\begin{cases} L_0 < x < L_1 : & \lambda \dfrac{\mathrm{d}^2 U(x)}{\mathrm{d}x^2} - kU(x) - \dfrac{2}{L_1 - b}\left(kT_B - kT\left(x\right) + Q_m\right) = 0 \\ x = L_0 : & W(L_0) = -\dfrac{q_0}{L_1 - b} \\ x = L_1 : & U(L_1) = 0 \end{cases} \tag{18}$$

3. Boundary element method

In order to solve the basic and additional problems the boundary element method has been applied. So, we consider the following equation

$$\frac{\mathrm{d}^2 F(x)}{\mathrm{d} x^2} - \omega^2 F(x) + \frac{1}{\lambda} S(x) = 0 \tag{19}$$

where $\omega^2 = k/\lambda$ and $F(x) = T(x)$, $S(x) = kT_B + Q_m$ for the basic problem (c.f. equation (1)), while $F(x) = 1U(x)$, $S(x) = -2[kT_B + Q_m - k\ T(x)]/(L_1 - b)$ for the additional problem (c.f. equation (18)).
The adequate BEM equation takes a form [5]

$$F(\xi) = \left[\frac{1}{\lambda} J^*(\xi, x) F(x) - \frac{1}{\lambda} F^*(\xi, x) J(x)\right]_{L_0}^{L_1} + \frac{1}{\lambda}\int_{L_0}^{L_1} S(x) F^*(\xi, x)\,\mathrm{d}x \tag{20}$$

where $\xi \in (L_0, L_1)$ is the observation point and

$$F^*(\xi, x) = \frac{1}{2\omega}\exp\left(-\omega|x-\xi|\right) \tag{21}$$

is the fundamental solution, while

$$J(x) = -\lambda\frac{\mathrm{d}\,F(x)}{\mathrm{d}x}, \quad J^*(\xi, x) = -\lambda\frac{\partial\,F^*(\xi, x)}{\partial x} \tag{22}$$

are the heat fluxes.
For $\xi \to L_0^+$ and $\xi \to L_1^-$ one obtains the system of equations

$$\begin{bmatrix} -\frac{1}{\lambda} F^*(L_0, L_0) & \frac{1}{\lambda} F^*(L_0, L_1) \\ -\frac{1}{\lambda} F^*(L_1, L_0) & \frac{1}{\lambda} F^*(L_1, L_1) \end{bmatrix} \begin{bmatrix} J(L_0) \\ J(L_1) \end{bmatrix} =$$

$$\begin{bmatrix} -\frac{1}{\lambda} J^*(L_0^+, L_0) - 1 & \frac{1}{\lambda} J^*(L_0^+, L_1) \\ -\frac{1}{\lambda} J^*(L_1^-, L_0) & \frac{1}{\lambda} J^*(L_1^-, L_1) - 1 \end{bmatrix} \begin{bmatrix} F(L_0) \\ F(L_1) \end{bmatrix} + \begin{bmatrix} \frac{1}{\lambda}\int_{L_0}^{L_1} S(x) F^*(L_0, x)\,\mathrm{d}x \\ \frac{1}{\lambda}\int_{L_0}^{L_1} S(x) F^*(L_1, x)\,\mathrm{d}x \end{bmatrix} \tag{23}$$

which allows to find the boundary values $F(L_0)$ and $J(L_1)$.
In the second stage of computations the values of function F at the internal points $\xi \in (L_0, L_1)$ are found using the formula (20).

4. Example of computations

The biological tissue of thickness $L_1 - L_0 = 0.035$ m has been considered. The thermal conductivity equals $\lambda = 0.75$ W/mK, perfusion rate: $G_B = 0.00125$ 1/s, volumetric specific heat of blood: $c_B = 3.9942 \cdot 10^6$ J/m^3 K, blood temperature: $T_B = 37$°C. The computations have been done for two different values of metabolic heat source, this means for $Q_m = 245$ W/m^3 (rest conditions) and $Q_m = 24500$ W/m^3 (exercise conditions). On the surface being in thermal contact with the external heat source ($x = L_0 = 0$) the heat flux $q_0 = 500$ W/m^2 has been assumed, while for internal surface of tissue ($x = L_1$) the temperature $T_b = 37$°C has been accepted.

It should be pointed out that the basic and additional problems can be solved analytically. In Figure 1 the temperature distribution obtained by the BEM (symbols) and in analytical way (full lines) is shown. The additional problem connected with the sensitivity analysis with respect to the shape parameter $b = L_0$ has been solved under the assumption that $b = 0$. Figure 2 shows the distribution of function U (full lines - analytical solution, symbols - BEM solution).

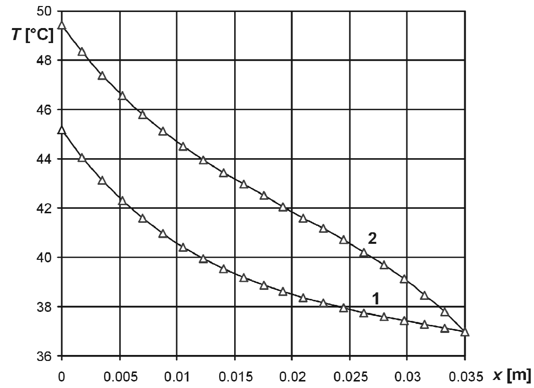

Fig. 1. Temperature distribution in tissue (1 - $Q_m = 245$, 2 - $Q_m = 24500$)

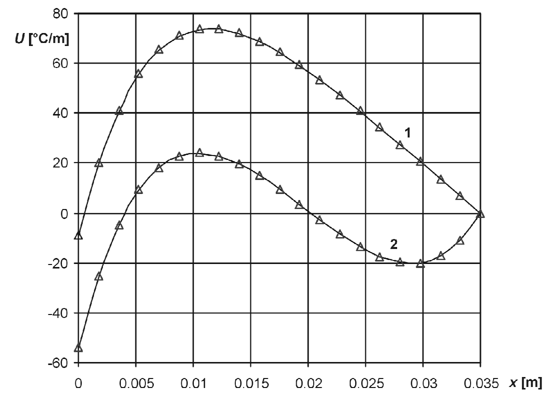

Fig. 2. Distribution of function $U = \mathrm{D}T/\mathrm{D}b$ (1 - $Q_m = 245$, 2 - $Q_m = 24500$)

Using the sensitivity function U we can estimate the change of temperature due to a change of the parameter b. So, we expand the function T into a Taylor series taking into account two component, namely

$$T(x, b + \Delta b) = T(x, b) + U(x, b)\Delta b \tag{24}$$

where Δb is the change of parameter b. In Figure 3 the solution for $\Delta b = 0.0035$ is presented (symbols - on the basic of formula (24), full lines - direct solution for $L_0 = 0.0035$ m).

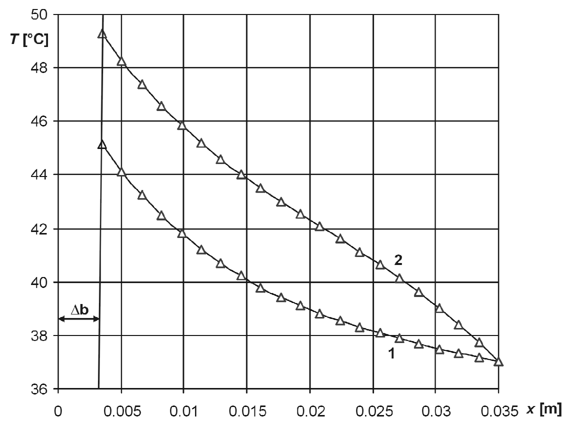

Fig. 3. Temperature distribution $T(x, b+\Delta b)$ (1 - $Q_m = 245$, 2 - $Q_m = 24500$)

Summing up, the shape sensitivity analysis allows, among others, to estimate the changes of temperature in the case when the geometry of the domain considered is changed.

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