ON A CERTAIN BOUNDARY VALUE PROBLEM FOR A LAMINATED HALF - SPACE

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Abstract. In this contribution we are to show that on the boundaries of a laminated medium, which are perpendicular to the laminae interfaces, we observe some near-boundary phenomena related to the oscillatory character of the boundary tractions. The aim of this note is to describe and discuss those phenomena in dynamic problems. The analysis is carried out in the framework of the tolerance averaging technique. General considerations are illustrated by a certain special problem describing decay of the traction oscillations.

1. Introduction

The object of considerations is a linear-elastic periodically laminated half-space. The boundary of the half-space is assumed to be perpendicular to the laminae interfaces. We are to show that in this case we deal with a certain near boundary effect caused by the periodic structure of the medium. In order to investigate this effect we shall apply the tolerance averaging technique [2], which makes it possible to investigate the effect of the period length on the macroscopic behaviour of a laminated solid. It has to be remembered that the effect under consideration cannot be described in the framework of the well-known homogenization procedure [1]. In this note the tolerance averaging technique will be presented in a certain simplified manner. The main new result of this contribution is a formulation of a certain boundary effect equation. This equation is a starting point for detailed analysis of the problem under consideration.

2. Preliminaries

To make this note self-consistent we shall outline below the modelling approach based on the tolerance averaging technique and leading to a certain macroscopic model of a laminated medium. However, the procedure constitutes a certain modification of that given in [2]. Analysis will be carried out in a Cartesian orthogonal coordinate system $Ox_1x_2x_3$ with $x_1 \geq 0$ as a laminated half-space, and a periodic structure in the Ox_3-axis direction. For the sake of simplicity we assume that the problem is plane and independent of the x_2-coordinate. By t we denote a time coordinate. Partial differentiation will be denoted by comma and time differentiation by overdot. The considerations will be restricted to a laminated medium with two kinds of laminae having thicknesses l' and l''; hence $l = l' + l''$ is an inhomogeneity period length. Recalling the general procedure discussed in [2] we shall use the averaging operator

$$\langle f \rangle(x) = \frac{1}{l} \int_{x-l/2}^{x+l/2} (y)\mathrm{d}y$$

where $x \equiv x_3$ and f is an arbitrary integrable function. Function f can also depend on arguments x_1, t as parameters. Obviously, if f is l-periodic function (i.e. function with a period l) then $\langle f \rangle = const$. Define $\mathbf{U} \equiv \langle \mathbf{u} \rangle$, where $\mathbf{u}(\cdot)$ is a displacement field. As a basic kinematic assumption we introduce the decomposition of the displacement

$$\mathbf{u}(x_1, x_3, t) = \mathbf{U}(x_1, x_3, t) + g(x_3)\mathbf{V}(x_1, x_3, t) \tag{1}$$

where $g(\cdot)$ is l-periodic continuous function (termed a shape function) such that:

$$g(0) = g(l) = l\sqrt{3}$$
$$g(l') = -l\sqrt{3}$$

and linear in $(0, l')$ and (l', l). Functions $\mathbf{U}(\cdot)$, $\mathbf{V}(\cdot)$ are new kinematic unknowns which are assumed to be slowly varying together with all their derivatives as functions of the x_3-coordinate. We recall that F is a slowly varying function of an argument x_3 provided that the following tolerance averaging formula holds

$$\langle fF \rangle(x_1, x_3, t) \cong \langle f \rangle(x_1, x_3, t) F(x_1, x_3, t) \tag{2}$$

in which $f(\cdot)$ is an arbitrary integrable function of x_3.

Let $\mathbf{T}$ and $\mathbf{E}$ stand for the stress and strain tensors, respectively. We shall assume that every material plane $x_3 = const$ is an elastic symmetry plane. Hence the stress-strain relations for a laminated medium under consideration will be assumed in the well-known form

$$\mathbf{T} = \mathbb{C} : \mathbf{E} \tag{3}$$

where $\mathbb{C}$ is a tensor of elastic modulae. In the problem under consideration $\mathbb{C}$ is l-periodic function of x_3. Under denotations $\nabla = (\partial_1, \partial_3)$ and $\partial = (\partial_1, 0)$ from assumption (1) and the strain - displacement relation $\mathbf{E} = (\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}})/2$ we obtain

$$\nabla\mathbf{u} \cong \nabla\mathbf{U} + \nabla g\mathbf{V} + g\partial\mathbf{V} \tag{4}$$

In order to formulate the averaged form of equations for $\mathbf{U}$, $\mathbf{V}$ we shall use the equations of motion

$$\rho\ddot{\mathbf{u}} - \nabla\mathbf{T} = 0 \tag{5}$$

where ρ is a mass density (which is l-periodic functions of x_3). In (5) we have neglected the effect of body forces on the dynamic behaviour of the laminated medium. Averaging the above equation and averaging the product of this equation with a shape function g, from Equations (1)-(4) we obtain:

$$\begin{aligned}&\langle\rho\rangle\ddot{\mathbf{U}} - \nabla\cdot(\langle\mathbb{C}\rangle:\nabla\mathbf{U} + \langle\mathbb{C}\cdot\nabla g\rangle\cdot\mathbf{V}) = \mathbf{0}\\&\langle\rho(g)^2\rangle\ddot{\mathbf{V}} + \langle\nabla g\cdot\mathbb{C}\cdot\nabla g\rangle\cdot\mathbf{V} + \partial(\langle(g)^2\mathbb{C}\rangle:\partial\mathbf{V}) - \langle\nabla g\cdot\mathbb{C}\rangle:\nabla\mathbf{U} = \mathbf{0}\end{aligned}\tag{6}$$

Equations (6) constitute a system of the second order partial differential equations for the basic unknowns **U**, **V** which have to be satisfied in a half space $x_1 > 0$. It can be seen that all coefficients in this system are constant. That is why the above system represents a certain macroscopic model of a laminated medium under consideration. Eqs (6) were derived in [2] by using an alternative procedure; a detailed discussion of these equations can be found in [2]. It has to be emphasized that Eqs (6) have a physical sense only if unknowns **U**, **V** together with their derivatives are slowly varying functions of an argument x_3 in the meaning of formula (2). Moreover, the underlined terms in (6) describe the effect of the period length on the macroscopic behaviour of the laminate; this effect is not taken into account if we deal with a homogenized model of a laminate [1]. On the other hand, Eqs (6) neglect certain terms which were derived by using the effective stiffness method of modelling [3]. However, these terms have not an important meaning for a solution to the initial boundary value problems for a medium under considerations.

3. Boundary conditions

Equation (6) will be considered together with the stress boundary conditions on $x_1 = 0$. Let $\mathbf{p}(x_3,t)$ be a traction at the half-space boundary and $\mathbf{n} = (-1,0)$ be a unit vector normal to the boundary of a laminated medium. Hence

$$\mathbf{p} = \mathbf{T}\cdot\mathbf{n} = [\mathbb{C}:\nabla\mathbf{U} + (\mathbb{C}\cdot\nabla g)\cdot\mathbf{V} + g\mathbb{C}:\partial\mathbf{V}]\cdot\mathbf{n}$$

The class of boundary tractions will be restricted to the form

$$\mathbf{p}(x_3,t) = \mathbf{p}_0(x_3,t) + r(x_3)\mathbf{p}_1(x_3,t)\tag{7}$$

where $r(\cdot)$ is a certain l-periodic function such that $\langle r\rangle = 0$ and $\mathbf{p}_0(\cdot,t)$, $\mathbf{p}_1(\cdot,t)$ are slowly varying functions. Hence $\langle\mathbf{p}\rangle \cong \mathbf{p}_0$, $\langle\mathbf{p}r\rangle \cong \mathbf{p}_1\langle r^2\rangle$ and the stress boundary conditions for equations (6) can be assumed in the form

$$\begin{aligned}&\mathbf{p}_0 = [\langle\mathbb{C}\rangle:\nabla\mathbf{U} + \langle\mathbb{C}\cdot\nabla g\rangle\cdot\mathbf{V}]\cdot\mathbf{n}\\&\mathbf{p}_1\langle r^2\rangle = [\langle r\mathbb{C}\rangle:\nabla\mathbf{U} + \langle r\mathbb{C}\cdot\nabla g\rangle\cdot\mathbf{V} + \langle rg\mathbb{C}\rangle:\partial\mathbf{V}]\cdot\mathbf{n}\end{aligned}\tag{8}$$

In applications function r can be assumed as either $r = g'$ or $r = g$. An alternative form of the stress boundary conditions can be also obtained by assuming that r is a periodic vector function defined by $r = \nabla g$; in this case $\mathbf{p}_1$ is a scalar function representing share tractions.

4. Boundary effect equation

In order to describe the effect of the boundary tractions on the macroscopic behaviour of a laminated half-space we shall formulate a certain approximate form of equations (6) and boundary conditions (8). To this and let us decompose vector field $\mathbf{V}$, setting $\mathbf{V} = \mathbf{V}_0 + \mathbf{V}_1 + \mathbf{V}_2$. Function $\mathbf{V}_0$ is assumed to satisfy equation

$$\langle \nabla g \cdot \mathbb{C} \cdot \nabla g \rangle \cdot \mathbf{V}_0 = -\langle \nabla g \cdot \mathbb{C} \rangle : \nabla \mathbf{U}$$

Solution to this equation takes the form

$$\mathbf{V}_0 = -\mathbf{K} : \langle \nabla g \cdot \mathbb{C} \rangle : \nabla \mathbf{U}$$

where $\mathbf{K}$ is the inverse to $\langle \nabla g \cdot \mathbb{C} \cdot \nabla g \rangle$. It follows that the first from Eqs (6) can be reduced to the form

$$\langle \rho \rangle \ddot{\mathbf{U}} - \nabla \cdot (\mathbb{C}^0 : \nabla \mathbf{U} + \langle \mathbb{C} \cdot \nabla g \rangle : \nabla (\mathbf{V}_1 + \mathbf{V}_2)) = \mathbf{0} \tag{9}$$

where $\mathbb{C}^0 = \langle \mathbb{C} \rangle - \langle \mathbb{C} \cdot \nabla g \rangle \cdot \mathbf{K} \cdot \langle \nabla g \cdot \mathbb{C} \rangle$ is a certain approximation of the known effective elasticity tensor. Equation (9) has to be considered together with the first stress boundary condition (8). Function $\mathbf{V}_1$ is assumed to satisfy the following equation

$$\langle \rho g^2 \rangle \ddot{\mathbf{V}}_1 + \partial \cdot (\langle g^2 \mathbb{C} \rangle \cdot \partial \mathbf{V}_1) - \langle \nabla g \cdot \mathbb{C} \cdot \nabla g \rangle \cdot \mathbf{V}_1 = \mathbf{0} \tag{10}$$

The above equation has to be considered together with the second from stress boundary conditions (8). At the same time for $\mathbf{V}_2$ we obtain

$$\langle \rho g^2 \rangle (\ddot{\mathbf{V}}_0 + \ddot{\mathbf{V}}_2) + \langle r \nabla g \cdot \mathbb{C} \cdot \nabla g \rangle \cdot \mathbf{V}_2 - \partial \cdot (\langle g^2 \mathbb{C} \rangle : \partial (\mathbf{V}_0 + \mathbf{V}_2)) = \mathbf{0} \tag{11}$$

together with homogenous boundary conditions. The idea of the proposed approximation is to neglect in (11) terms of an order $O(l^2)$. In this case we obtain $\mathbf{V}_2 \cong 0$. Thus, the proposed model equations reduce to the form

$$\langle \rho \rangle \ddot{\mathbf{U}} - \nabla \cdot (\mathbb{C}^0 : \nabla \mathbf{U}) = -\langle \mathbb{C} \cdot \nabla g \rangle : \nabla \mathbf{V}_1 \tag{12}$$

and

$$\langle \rho g^2 \rangle \ddot{\mathbf{V}}_1 + \partial \cdot (\langle g^2 \mathbb{C} \rangle : \partial \mathbf{V}_1) - \langle \nabla g \cdot \mathbb{C} \cdot \nabla g \rangle \cdot \mathbf{V}_1 = \mathbf{0} \tag{13}$$

Equation (12) has to be considered with the boundary conditions (8) for $\mathbf{p}_0$. Similarly, for equation (13) we have to take into account boundary conditions (8) for $\mathbf{p}_1$.

5. Illustrative example

So far, the form of the function r in formula (7) has not been specified. Now assume that the composition (7) is taken in the form

$$\mathbf{p} = \mathbf{p}_0 + \nabla g p_1 \tag{14}$$

where by virtue of $\nabla g = (0, g_{,3})$ term p_1 represents a scalar field which characterise the oscillating part of the boundary tractions which are tangent to the boundary $x_1 = 0$. We can show that under condition (14) the boundary effect Equation (13) together with the boundary condition (8) depends only on the function $\mathbf{V}_1$:

$$\begin{aligned} &\langle \rho g^2 \rangle \ddot{\mathbf{V}}_1 + \partial \cdot (\langle g^2 \mathbb{C} \rangle : \partial \mathbf{V}_1) - \langle \nabla g \cdot \mathbb{C} \cdot \nabla g \rangle \cdot \mathbf{V}_1 = \mathbf{0} \quad \text{for } x_1 > 0 \\ &\mathbf{p}_1 \langle (\nabla g)^2 \rangle = [\langle \nabla g \cdot \mathbb{C} \cdot \nabla g \rangle \mathbf{V}_1] \cdot \mathbf{n} \quad \text{for } x_1 = 0 \end{aligned} \tag{15}$$

The above relations can be also represented in the form:

$$\begin{aligned} &\left\langle \rho g^2 \right\rangle \ddot{\mathbf{V}}_1^3 + \left\langle g^2 \mathbf{C}_{1313} \right\rangle \mathbf{V}_1^3{}_{,11} - \left\langle (g_{,3})^2 \, \mathbf{C}_{1313} \right\rangle \cdot \mathbf{V}_1^3 = 0 \quad \text{for } x_1 > 0 \\ &p_1^3 \left\langle (g_{,3})^2 \right\rangle = -\left\langle (g_{,3})^2 \, \mathbf{C}_{1313} \right\rangle \cdot \mathbf{V}_1^3 \quad \text{for } x_1 = 0 \end{aligned} \tag{16}$$

Let as denote:

$$\mathbf{G} \equiv \mathbf{C}_{1313}, \quad \mathbf{G}_0 \equiv \left\langle (g_{,3})^2 \, \mathbf{C}_{1313} \right\rangle, \quad l^2 \mathbf{G}_1 \equiv \left\langle g^2 \mathbf{C}_{1313} \right\rangle$$

$$\overline{\mathbf{G}}_0 \equiv \left\langle (g_{,3})^2 \, \mathbf{C}_{1313} \right\rangle \Big/ \left\langle (g_{,3})^2 \right\rangle, \quad \psi \equiv \mathbf{V}_1^3, \quad \mathbf{s} \equiv p_1^3$$

Under the above notations the problem under consideration will be given by

$$\begin{aligned} &l^2 \left\langle \rho \right\rangle \ddot{\psi} + l^2 \mathbf{G}_1 \psi_{,11} - \mathbf{G}_0 \psi = 0 \quad \text{for } x_1 > 0 \\ &\mathbf{s} = -\overline{\mathbf{G}}_0 \psi \quad \text{for } x_1 = 0 \end{aligned} \tag{17}$$

The above boundary effect equation together with the boundary condition for $x_1 = 0$ have to be considered with the condition at the infinity $\psi \to 0$ when $x_1 \to \infty$.

Now assume the boundary traction $\mathbf{s}$ is harmonic in time and hence will be assumed in the form $\mathbf{s} = \overline{\mathbf{s}}(x_3)\cos\omega t$. Thus we can look for the solution ψ of Equation (17) in the form $\psi = \overline{\psi}(x_1, x_3)\cos\omega t$ where for $\overline{\psi}$ we obtain equation:

$$\begin{aligned} &l^2 \mathbf{G}_1 \overline{\psi}_{,11} - \left(\mathbf{G}_0 + l^2 \omega^2 \left\langle \rho \right\rangle \right) \overline{\psi} = 0 \\ &\overline{\mathbf{s}}(x_3) = -\overline{\mathbf{G}}_0 \psi(0, x_3) \quad \text{for } x_1 = 0 \end{aligned} \tag{18}$$

Under denotations $\Omega^2 = \dfrac{\langle \rho \rangle}{\mathbf{G}_1}\omega^2$, $\mathbf{K}^2 = \dfrac{\mathbf{G}_0}{\mathbf{G}_1}$ we obtain

$$\overline{\psi}_{,11} - \left(\frac{\mathbf{K}^2}{l^2} + \Omega^2 \right)\overline{\psi} = 0 \tag{19}$$

Taking into account conditions at the infinity we obtain finally

$$\overline{\psi}(x_1, x_3) = -\frac{\overline{\mathbf{s}}(x_3)}{\mathbf{G}_0}\exp\left(-x_1\sqrt{\frac{\mathbf{K}^2}{l^2} + \Omega^2} \right) \quad \text{for } x_1 \ge 0,\ x_3 \in (-\infty, \infty) \tag{20}$$

Formula (20) represents a solution to the boundary effect equation in the problem under consideration. Using this formula we can analyse the form of the decay function ψ as a function of dimensionless vibration frequency Ω and the period length. The particulars related to the above assertion will be discussed in a separate contribution.

6. Conclusions

The main conclusions is that Equation (13) involves only one unknown function $\mathbf{V}_1$. It can be shown that this equation together with an appropriate boundary condition describes the effect of the traction disturbances $\mathbf{p}_1$ on the dynamic macroscopic solid behaviour. It follows that equation (13) can be referred to as the boundary effect equation since function $\mathbf{V}_1$ describes the disturbances of the displacement field according to the decomposition (1). On the other hand, Equation (12) describes the solid behaviour independent on the boundary traction disturbances.

References

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