FREE LONGITUDINAL VIBRATIONS OF MULTI-STEP NON-UNIFORM RODS

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Abstract. The paper presents the solution of the free longitudinal vibration problem for a multi-step rod with varying cross-section areas. The problem considered takes into account vibrations of rods with additional discrete spring-mass elements attached at their ends. The problem solution is obtained with the application of the Green's function properties and the power series method. The presented numerical example shows the influence of the selected parameter characterizing the discrete element on free vibration frequencies of the system.

1. Introduction

The vibration analysis of non-uniform rods is the subject of papers [1-6]. The solution of the vibration problem in a closed form can be found in particular cases of the cross-sectional area and the mass per unit length of the rod. It is often assumed that the mass is constant and the cross-sectional area varying with respect to the space variable. In this case the solution of the free vibration problem is well known for katenoidal, sinusoidal and exponentional rods [1]. For the rod with a cross-sectional area $A(\xi) = (\alpha\xi + \beta)^c$, where c is a real number, a solution has been presented by Kumar [2] and Li [3]. The free vibration problem of a system of non-uniform rods coupled by springs was the subject of paper [5]. Such problems can be solved with the application of the Green's function method.

The Green's function method was used in paper [7] to solve the vibration problem of a system of two uniform rods coupled by translational springs. The frequency equation and mode shapes are expressed by Green's functions corresponding to rods which are components of the considered combined system. The application of the method is possible only if the right Green's functions are known. A solution of the free vibration problem for non-uniform rods carrying an arbitrary number of discrete elements was presented in reference [6].

In this paper, the Green's function method is used to solve the free vibration problem of a rod consisting of any number of non-uniform segments. The necessary Green's functions have been derived with the use of a power series method. An example of numerical calculations of free longitudinal vibration frequencies of a multi-step rod is given.

2. Formulation and solution of the problem

Consider a rod consisting of n non-uniform segments as shown in Figure 1. Assume that a discrete spring-mass element is attached at point $x = L$. Free vibration of the rod is governed by the following differential equations:

$$\mathbf{L}_1\left[u_1\left(x,t\right)\right] = s_1(t)\delta(x-L_1) \quad \text{for } x \in [0,L_1] \tag{1}$$

$$\mathbf{L}_i\left[u_i\left(x,t\right)\right] = -s_{i-1}(t)\delta(x-L_{i-1}) + s_i(t)\delta(x-L_i) \;\; x \in [L_{i-1},L_i],\; i = 2,\dots, n-1 \tag{2}$$

$$\mathbf{L}_n\left[u_n\left(x,t\right)\right] = -s_{n-1}(t)\delta(x-L_{n-1}) + p(x,t) \quad \text{for } x \in [L_{n-1},L] \tag{3}$$

where $\mathbf{L}_i = \frac{\partial}{\partial x}[E\overline{A}_i(x)\frac{\partial}{\partial x}] - \rho\overline{A}_i(x)\frac{\partial^2}{\partial t^2}$ for i = 1,2,…, n, $\overline{A}_i\left(x\right)$ is the area of the cross-section at point x of the rod, E is the modulus of elasticity, ρ is the mass density of the rod material, $\delta(\ )$ denotes the Dirac delta function, $p(x,t) = \left[m\frac{\partial^2 u_n(x,t)}{\partial t^2} + ku_n(x,t)\right]\delta\left(x-L\right)$, m and k are the discrete mass and the stiffness coefficient of the spring attached at point $x = L$, respectively.
The functions u_i satisfy homogeneous boundary conditions which may be symbolically written in the following form:

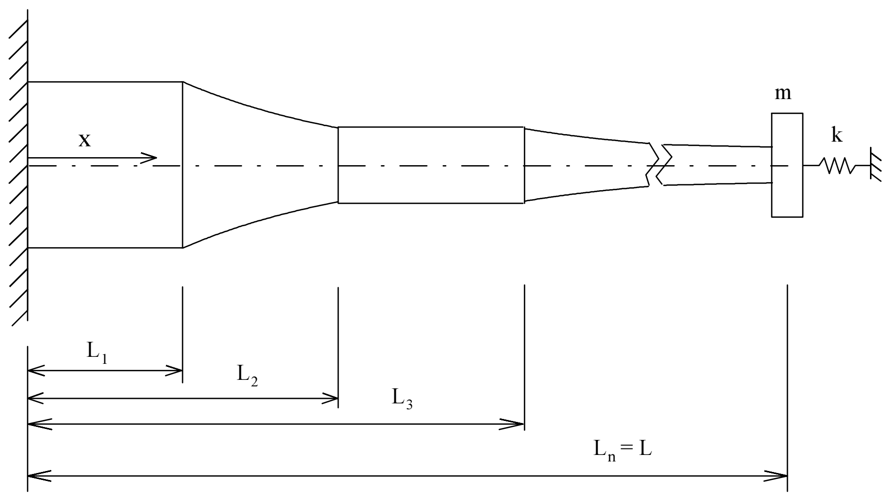

Fig. 1. A sketch of a multi-step non-uniform rod with a spring-mass discrete element

$$\overline{\mathbf{B}}_0[u_1]\big|_{x=0} = 0 \quad \overline{\mathbf{B}}_1[u_n]\big|_{x=L} = 0 \tag{4}$$

where $\overline{\mathbf{B}}_0$ and $\overline{\mathbf{B}}$ are linear, spatial differential operators and L denotes the length of the rod. Moreover, at points L_i the following continuity conditions are satisfied:

$$u_i(L_i,t) = u_{i+1}(L_i,t), \quad i = 1,..., n-1 \tag{5}$$

Assuming that the natural frequencies of the rod are harmonic, i.e. substituting the functions:

$$u_i(x,t) = U_i(x)e^{j\omega t}, \quad p(x,t) = P(x)e^{j\omega t}, \quad s_i(t) = \overline{S}_i e^{j\omega t} \tag{6}$$

into equations (1)-(3), one obtains:

$$\tilde{\mathbf{L}}_1[U_1] = S_1\delta(\xi - l_1) \quad \text{for } \xi \in [0,l_1] \tag{7}$$

$$\tilde{\mathbf{L}}_i[U_i] = -\mu_{i-1}S_{i-1}\delta(\xi) + S_i\delta(\xi - l_i) \quad \text{for } \xi \in [0,l_i],\ i = 2,..., n-1 \tag{8}$$

$$\tilde{\mathbf{L}}_n[U_n] = -\mu_{n-1}S_{n-1}\delta(\xi) + P(\xi) \quad \text{for } \xi \in [0,l_n] \tag{9}$$

where: $j^2 = -1$, $\Omega^2 = \rho\omega^2/E$, $\Lambda^2 = \rho\overline{\omega}^2/E$, $\overline{\omega} = \sqrt{k/m}$, $S_i = \overline{S}_i/E\overline{A}_i(0)$, ω are the natural frequencies of the rod, $\xi = x - L_{i-1}$, $l_i = L_i - L_{i-1}$, $\eta = m/\rho\overline{A}_n(0)$, $A_i(\xi) = \overline{A}_i(x)/\overline{A}_i(0)$, $P(\xi) = \eta(\Lambda^2 - \Omega^2)U_n(\xi)\delta(\xi - l_n)$, $\mu_{i-1} = \overline{A}_{i-1}(0)/\overline{A}_i(0)$, $\tilde{\mathbf{L}}_i = \frac{d}{d\xi}\left[A_i(\xi)\frac{d}{d\xi}\right] + \Omega_i^2 A_i(\xi)$ for $\xi \in [0,l_i]$, $i = 1,...n$.

The boundary and continuity conditions (4), (5) may be written in the following form:

$$\mathbf{B}_0[U_1]\big|_{\xi=0} = 0, \quad \mathbf{B}_1[U_n]\big|_{\xi=l_n} = 0 \tag{10}$$

$$U_i(l_i) = U_{i+1}(0), \quad i = 1,..., n-1 \tag{11}$$

The solution of problems (7)-(11) is obtained with the use of the properties of Green's functions G_i. If the Green's functions are known, then the following relationships are obtained on the basis of equations (7)-(9):

$$U_1(\xi) = S_1G_1(\xi,l_1) \tag{12}$$

$$U_i(\xi) = -\mu_{i-1}S_{i-1}G_i(\xi,0) + S_iG_i(\xi,l_i) \quad \text{for } i = 2,...n-1 \tag{13}$$

$$U(\xi) = -\mu S G(\xi,0) + \eta(\Lambda - \Omega)U(l)G(\xi,l) \tag{14}$$

A system of equations is obtained by substituting functions $U_i(\xi)$ into continuity conditions (11):

$$S_1[G_1(l_1,l_1)+\mu_1 G_2(0,0)]-S_2G_2(0,l_2)=0 \tag{15}$$

$$-\mu_{i-1}S_{i-1}G_i(l_i,0)+S_i[G_i(l_i,l_i)+\mu_i G_{i+1}(0,0)]-S_{i+1}G_{i+1}(0,l_{i+1})=0,\ i=2,3,...n-2 \tag{16}$$

An additional equation is obtained by setting in equation (14) $\xi = l_n$:

$$\mu_{n-1}S_{n-1}G_n(l_n,0)+[1+\eta(\Omega^2-\Lambda^2)G_n(l_n,l_n)]U_n(l_n)=0 \tag{17}$$

The set of equations (15)-(17) may be written in the matrix form

$$\mathbf{A}(\omega)\ \mathbf{S}=\mathbf{0} \tag{18}$$

where $\mathbf{S} = [S_1, S_2,\dots, S_{n-1}, U_n(l_n)]^T$, $\mathbf{A}=[a_{ij}]$ is n x n dimensional matrix with $a_{i\,i-1}=-\mu_{i-1}G_i(l_i,0)$ for $i = 2,3,...,n$; $a_{ii}=G_i(l_i,l_i)+\mu_i G_{i+1}(0,0)$ for $i = 1,2,...,n-1$; $a_{i\,i+1}=-G_{i+1}(0,l_{i+1})$ for i = $1,2,...,n-2$; $a_{n-1\,n}=\eta(\Omega^2-\Lambda^2)G_n(0,l_n)$; $a_{nn}=1+$ $+\,\eta(\Omega^2-\Lambda^2)G_n(l_n,l_n)$ and the remaining coefficients a_{ij} are equal to zero.

For a non-trivial solution of the problem, the determinant of the coefficient matrix is set equal to zero, yielding the equation

$$\det \mathbf{A}(\omega) = 0 \tag{19}$$

Equation (19) (the equation of natural frequencies of the rod, with unknown eigenfrequencies ω) is then solved numerically.

3. The use of the power series method in deriving the Green's functions

The solution $U_i(\xi)$ of the eigenproblem (7)-(11) was obtained with the use of known Green's functions. The Green's functions G_i satisfy the equation:

$$\tilde{\mathbf{L}}_i\left[G_i(\xi,\zeta)\right]=\delta(\xi-\zeta),\quad i=1,2,\dots,n \tag{20}$$

Functions G_1, G_n satisfied the boundary conditions (10) and G_i - the conditions corresponding to the free ends of the rod, i.e.:

$$G_{i,\xi}\Big|_{\xi=0}=0,\ \text{for}\ i=2,3,...,n,\ \text{and}\ G_{i,\xi}\Big|_{\xi=l_i}=0,\ \text{for}\ i=1,2,...,n-1 \tag{21}$$

The function G_i may be written in the form

$$G_i(\xi,\zeta) = G_i^0(\xi,\zeta) + G_i^1(\xi,\zeta)H(\xi-\zeta) \tag{22}$$

where $G_i^0(\xi,\zeta)$ is a general solution of the homogeneous equation:

$$\tilde{\mathbf{L}}_i\left[G_i(\xi,\zeta)\right] = 0, \quad i = 1,2,\ldots,n \tag{23}$$

and $G_i^1(\xi,\zeta)H(\xi-\zeta)$ is a particular solution of equation (20). It may be proved that the functions $G_i^1(\xi,\zeta) = G_i^1(\xi-\zeta)$ are solutions of equation (23) which satisfy the following conditions:

$$G_i^1\Big|_{\xi=\zeta} = 0, \quad \frac{dG_i^1}{d\xi}\Bigg|_{\xi=\zeta} = \frac{1}{A_i(\zeta)} \tag{24}$$

The general solution $V(\xi)$ of the differential equation $\tilde{\mathbf{L}}\left[V(\xi)\right] = 0$ will be determined assuming that function $A_i(\xi)$ is expressed as (the index *i* is omitted)

$$A(\xi) = \sum_{r=0}^{\infty} \frac{a_r}{r!}\xi^r \tag{25}$$

The function $V(\xi)$ is searched with the use of the power series method

$$V(\xi) = \sum_{r=0}^{\infty} \frac{\upsilon_r}{r!}\xi^r \tag{26}$$

Substituting functions $A(\xi)$ and $V(\xi)$ into equation (23), one obtains:

$$\sum_{j=0}^{r+1}\binom{r+1}{j}\upsilon_{j+1}a_{r+1-j} + \Omega^2\sum_{j=0}^{r}\binom{r}{j}\upsilon_j a_{r-j} = 0, \quad r = 0,1,2,... \tag{27}$$

Finally, one obtains the following formula as a result of equation (27):

$$\upsilon_{r+2} = \frac{-1}{a_0}\left[\sum_{j=0}^{r}\binom{r+1}{j}\upsilon_{j+1}a_{r+1-j} + \Omega^2\sum_{j=0}^{r}\binom{r}{j}\upsilon_j a_{r-j}\right], \quad r = 0,1,2,... \tag{28}$$

As each of coefficients υ_r is a combination of coefficients υ_0 and υ_1, the function *U* may be written in the form

$$V(\xi) = \upsilon_0 P(\xi) + \upsilon_1 Q(\xi) \tag{29}$$

where: $P(\xi) = \sum_{r=0}^{\infty} p_r\xi^r$, $Q(\xi) = \sum_{r=0}^{\infty} q_r\xi^r$.

Taking into account the conditions (24) in equation (22), one obtains:

$$u_{1,0}(\zeta) = -Q(\zeta)/w_0(\zeta,\zeta)A(\zeta), \quad u_{1,1}(\zeta) = P(\zeta)/w_0(\zeta,\zeta)A(\zeta) \tag{30}$$

where $w_0(\xi,\zeta) = P(\xi)Q'(\zeta) - Q(\xi)P'(\zeta)$. Finally, the function $G(\xi,\zeta)$ can be written in the form

$$G(\xi,\zeta) = u_{0,0}(\zeta)P(\xi) + u_{0,1}(\zeta)Q(\xi) + R(\xi,\zeta)H(\xi-\zeta)/W(\zeta) \tag{31}$$

where: $W(\zeta) = -1/w_0(\zeta,\zeta)A(\zeta)$ and $R(\xi,\zeta) = P(\xi)Q(\zeta) - P(\zeta)Q(\xi)$.
The coefficients $u_{0,0}(\zeta)$ and $u_{0,1}(\zeta)$ are determined with the use of boundary conditions. For example, the Green's functions for clamped-free and free-free rods are the following:

- a clamped-free rod ($G\big|_{\xi=0} = 0,\ G,_{\xi}\big|_{\xi=l} = 0$)

$$G(\xi,\zeta) = W(\zeta)\left[w_0(l,\zeta)Q(\xi)/Q'(l) + R(\xi,\zeta)H(\xi-\zeta)\right] \tag{32}$$

- a free-free rod ($G,_{\xi}\big|_{\xi=0} = 0,\ G,_{\xi}\big|_{\xi=l} = 0$)

$$G(\xi,\zeta) = W(\zeta)\left[w_0(l,\zeta)P(\xi)/P'(l) + R(\xi,\zeta)H(\xi-\zeta)\right] \tag{33}$$

5. Numerical example

Let us consider a rod consisting of two segments. First of them is a prismatic rod with the length l_1 and the cross-section area of the second one (with length l_2) is expressed as $A_2(\xi) = (\xi+1)^4$. The rod is clamped at the end $\xi = 0$ and free at $\xi = L$.
The frequency equation (19) for this rod is as follows

$$\left[G_1(l_1,l_1) + \mu_1 G_2(0,0)\right]\left[1 + \eta(\Omega^2 - \Lambda^2)G_2(l_2,l_2)\right] - \mu_1\eta(\Omega^2 - \Lambda^2)G_2(l_2,0)G_2(0,l_2) = 0 \tag{34}$$

where the function $G_2(\xi,\zeta)$ is given by formula (33) for $i = 2$ and $G_1(\xi,\zeta)$ has the form

$$G_1(\xi,\zeta) = -\Omega^{-1}\left[\cos\Omega(l_1-\zeta)\sin\Omega\xi/\cos\Omega - \sin\Omega(\xi-\zeta)H(\xi-\zeta)\right] \tag{35}$$

The calculations are performed for various values of the parameter α which characterizes the non-uniformity of the second segment of the rod.

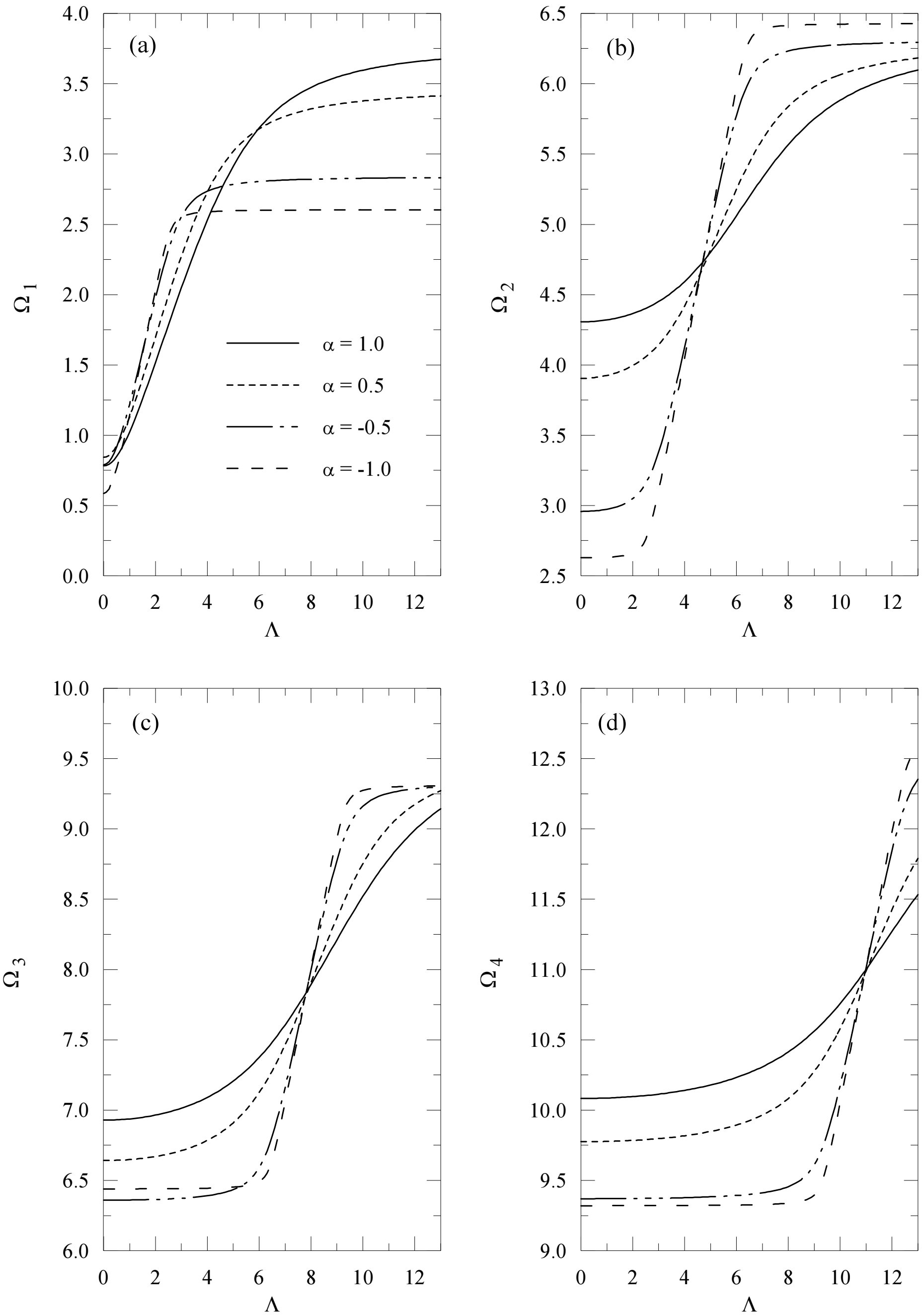

Fig. 2. Frequency parameter values Ω_i for the first four modes of vibration as a function of $\Lambda = \varpi\sqrt{\rho/E}$ for a clamped-free rod with a cross-section area $A_2(\xi) = (\alpha\xi + 1)^4$

A discrete spring-mass element is attached at the free end of the rod. The free vibration frequency of the isolated spring-mass system is determined by the parameter $\Lambda = \bar{\omega}\sqrt{\rho/E}$. Natural frequencies of the free longitudinal vibration Ω_i (i = 1,..,4) are numerically calculated as functions of the parameter Λ and they are presented in Figure 2. The results show that an increase of the frequency of the isolated spring-mass system causes an increase of free vibration frequencies of the compound system. It is interesting that in case of second and higher modes there are such Λ, for which the values of the Ω_i do not depend on the non-uniformity parameter α (the intersection points of the curves on Figures 2(a)-(d)).

6. Conclusions

The solution for free longitudinal vibrations of a rod consisting of n non-uniform segments was obtained with the use of the power series method and the Green's function properties. The presented numerical example has shown the influence of the parameter characterizing the attached discrete element on free vibration frequencies of the system. Although the presented numerical example deals with non-uniform rods consisting of two segments, the solution can be used for a vibration analysis of rods consisting of an arbitrary number of segments.

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