THE SETS OF CERTAIN CLASSES IN GENERALIZED METRIC SPACES

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Abstract. In this paper some property of sets of certain classes in the generalized metric spaces are considered. In last section of this paper an example of a certain set of these classes in two-dimensional Euklidean space will be given.

1. Introduction

Let E be a certain non-empty set and let l be any non-negative real function defined on the Cartesian product $E_0 \times E_0$ of the family E_0 of all non-empty subsets of the set E. The pair (E, l) we shall call the generalized metric space.

Let a, b be arbitrary non-negative real functions defined in a certain right-hand side neighbourhood of 0 such that

$$a(r) \xrightarrow[r \to 0^+]{} 0 \quad \text{and} \quad b(r) \xrightarrow[r \to 0^+]{} 0 \tag{1}$$

By $S_l(p, r)_{a(r)}$ and $S_l(p, r)_{b(r)}$ we denote in this paper so-called $a(r)$, $b(r)$-neighbourhoods of the sphere $S_l(p, r)$ with the centre at the point p and the radius r in the space (E, l).

We say that the pair (A, B) of sets of the family E_0 is (a, b)-clustered at the point p of the space (E, l), if 0 is the cluster point of the set of all numbers $r > 0$ such that $A \cap S_l(p, r)_{a(r)} \neq \emptyset$ and $B \cap S_l(p, r)_{b(r)} \neq \emptyset$.

Let k be any, but fixed positive real number, and let by the definition (see the paper [9]):

$$T_l(a, b, k, p) = \{(A, B) : \ A, B \in E_0, \text{ the pair } (A, B) \text{ is } (a, b)\text{-clustered at the point } p \text{ of the space } (E, l) \text{ and } \frac{1}{r^k} l(A \cap S_l(p, r)_{a(r)}, B \cap S_l(p, r)_{b(r)}) \xrightarrow[r \to 0^+]{} 0\} \tag{2}$$

The set $T_l(a, b, k, p)$ defined by the formula (2) we call the relation of (a, b)-tangency of order k at the point p (shortly: the tangency relation) of sets in the generalized metric space (E, l).

If $(A,B) \in T_l(a,b,k,p)$, then we say that the set $A \in E_0$ is (a,b)-tangent of order k to the set $B \in E_0$ at the point p of the space (E,l).

We say (see [3]) that the set $A \in E_0$ has the Darboux property at the point p of the generalized metric space (E,l), and we shall write this as: $A \in D_p(E,l)$, if there exists a number $\tau > 0$ such that $A \cap S_l(p,r) \neq \emptyset$ for $r \in (0,\tau)$.

In this paper we shall consider certain problems concerning the tangency of sets of the classes $\widetilde{M}_{p,k}$ having the Darboux property at the point p of the generalized metric spaces (E,l), for $l \in \mathfrak{F}_f$. A certain theorem for the sets of these classes will be given here.

2. On a certain theorem

Let ρ be an arbitrary metric of the set E. We shall denote by $d_\rho A$ the diameter of the set $A \in E_0$, and by $\rho(A,B)$ the distance of sets $A,B \in E_0$ in the metric space (E,ρ).

Let f be any subadditive increasing real function defined in a certain right--hand side neighbourhood of 0, such that $f(0)=0$.
By $\mathfrak{F}_f$ we will denote the class of all functions l fulfilling the conditions:

1^0 $\quad l : E_0 \times E_0 \longrightarrow \langle 0,\infty)$,

2^0 $\quad f(\rho(A,B)) \leq l(A,B) \leq f(d_\rho(A \cup B)) \quad$ for $\quad A,B \in E_0$.

It is easy to check that every function $l \in \mathfrak{F}_f$ generates in the set E the metric l_0 defined by the formula:

$$l_0(x,y) = f(\rho(x,y)) \quad \text{for} \quad x,y \in E \tag{3}$$

Let us put by definition (see [6])

$$\begin{aligned} \widetilde{M}_{p,k} = \{A \in E_0 :\ & p \in A' \text{ and there exists } \mu > 0 \text{ such that} \\ & \text{for an arbitrary } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that} \\ & \text{for every pair of points } (x,y) \in [A,p;\mu,k] \\ & \text{if } \rho(p,x) < \delta \text{ and } \frac{\rho(x,A)}{\rho^k(p,x)} < \delta, \text{ then } \frac{\rho(x,y)}{\rho^k(p,x)} < \varepsilon\} \end{aligned} \tag{4}$$

where A' is the set of all cluster points of the set $A \in E_0$ and

$$[A,p;\mu,k] = \{(x,y) : x \in E,\ y \in A \text{ and } \mu\rho(x,A) < \rho^k(p,x) = \rho^k(p,y)\} \tag{5}$$

Theorem 1. *If the set $A \in E_0$ is (a,b)-tangent of order k to the set $B \in E_0$ at the point $p \in E$ for an arbitrary function $l \in \mathfrak{F}_f$ and for every point x such* *that* $(x,y) \in [A,p;\mu,k]$ *there exists a point* $\widetilde{y} \in A \cap S_l(p,r)_{a(r)}$ *and* $\lambda > 0$ *such that*

$$\rho(x,\widetilde{y}) \leq \lambda \rho(x,A) \tag{6}$$

then A *is the set of the class* $\widetilde{M}_{p,k}$.

Proof. Let $(A,B) \in T_l(a,b,k,p)$ for $l \in \mathfrak{F}_f$ and $A,B \in E_0$. From here, in particular, it follows that

$$(A,B) \in T_l(a,b,k,p) \quad \text{for} \quad l \in \mathfrak{F}_{id} \quad \text{and} \quad A,B \in E_0 \tag{7}$$

where id denotes the identity function defined in a certain right-hand side neighbouhood of 0. Because every function $l \in \mathfrak{F}_{id}$ generates in the set E the metric ρ (see definition of the class $\mathfrak{F}_f$), then from here and from (7) follows

$$\frac{1}{r^k} l(A \cap S_l(p,r)_{a(r)}, B \cap S_l(p,r)_{b(r)}) \xrightarrow[r \to 0^+]{} 0 \tag{8}$$

Putting $l = d_\rho$, from (8) we get

$$\frac{1}{r^k} d_\rho((A \cap S_l(p,r)_{a(r)}) \cup (B \cap S_l(p,r)_{b(r)})) \xrightarrow[r \to 0^+]{} 0 \tag{9}$$

Because

$$d_\rho(A \cap S_l(p,r)_{a(r)}) \leq d_\rho((A \cap S_l(p,r)_{a(r)}) \cup (B \cap S_l(p,r)_{b(r)}))$$

then from here, from (9) we obtain

$$\frac{1}{r^k} d_\rho(A \cap S_l(p,r)_{a(r)}) \xrightarrow[r \to 0^+]{} 0 \tag{10}$$

From (10) it follows that for an arbitrary $\varepsilon > 0$ there exists $\delta_1 > 0$ such that

$$\frac{1}{r^k} d_\rho(A \cap S_\rho(p,r)_{a(r)}) < \frac{\varepsilon}{2} \quad \text{for} \quad 0 < r < \delta_1 \tag{11}$$

Now we shall prove that for every pair of points (x,y) of the set $[A,p;\mu,k]$

$$\frac{\rho(x,y)}{\rho^k(p,x)} < \varepsilon \tag{12}$$

if only

$$r = \rho(p,x) < \delta \quad \text{and} \quad \frac{\rho(x,A)}{\rho^k(p,x)} < \delta \tag{13}$$

Let us put $\mu = 1$ and $\delta = \min(1, \frac{\varepsilon}{2\lambda}, \delta_1)$. From here, from (6), (11) and from the triangle inequality we have

$$\frac{\rho(x,y)}{\rho^k(p,x)} \le \frac{\rho(x,\widetilde{y})}{\rho^k(p,x)} + \frac{\rho(\widetilde{y},y)}{\rho^k(p,x)} < \frac{1}{r^k} d_\rho(A \cap S_\rho(p,r)_{a(r)}) + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

what means that A is the set of the class $\widetilde{M}_{p,k}$.

3. On a certain set of the class $\widetilde{M}_{p,k}$

In this Section we will give an example of a certain set of the class $\widetilde{M}_{p,k}$ in two-dimensional Euklidean space, and will use Theorem 2.3 of the paper [7] for certain subsets of this set.

Example 1. Let $E = \mathbf{R}^2$ be the two-dimensional Euclidean space. Let φ be a increasing function of the class C_1 (homogenous function together with 1st derivative) defined in a certain right-hand side neighbourhood of 0 such that $\varphi(0) = 0$. Using the de L′Hospital′s theorem and mathematical induction for $k \in \mathbf{N}$ we can easily prove that

$$\frac{\varphi^{k+1}(t)}{t^k} \xrightarrow[t \to 0^+]{} 0 \tag{14}$$

From this it follows immediately

$$\frac{\varphi^{2k+2}(t)}{t^{2k}} \xrightarrow[t \to 0^+]{} 0 \tag{15}$$

Let us put

$$C = \{(x,y):\ x \ge 0,\ \ 0 \le y \le \varphi^{k+1}(x) \ \text{ and } \ k \in \mathbf{N}\} \tag{16}$$

We shall prove that C defined by the formula (16) is the set of the class $\widetilde{M}_{p,k}$, where $p = (0,0)$. For this purpose let us denote

$$A = \{(t,0):\ t \ge 0\} \quad \text{and} \quad B = \{(t,\varphi^{k+1}(t)):\ t \ge 0,\ k \in \mathbf{N}\} \tag{17}$$

Let $y_1,\ y_2$ be a points of the set C such that

$$y_1 \in A \cap S_\rho(p,r), \quad y_2 \in B \cap S_\rho(p,r) \quad \text{for} \quad r > 0 \tag{18}$$

If according to (17) and (18) we put $y_2 = (t, \varphi^{k+1}(t))$, then

$$r = \rho(p,y_2) = \sqrt{t^2 + \varphi^{2k+2}(t)} \tag{19}$$

Hence it follows that $y_1 = (\sqrt{t^2 + \varphi^{2k+2}(t)}, 0)$. From (19) and from the properties of the function φ it results also that $r \to 0^+$ if and only if $t \to 0^+$. Hence and from the conditions (14), (15), (19) for $r > 0$ we have

$$\begin{aligned}
\frac{1}{r^{2k}}\rho^2(y_1,y_2) &= \frac{(\sqrt{t^2+\varphi^{2k+2}(t)}-t)^2+\varphi^{2k+2}(t)}{(t^2+\varphi^{2k+2}(t))^k}\\
&= 2\frac{t^2+\varphi^{2k+2}(t)-t\sqrt{t^2+\varphi^{2k+2}(t)}}{(t^2+\varphi^{2k+2}(t))^k}\\
&= 2\frac{\varphi^{2k+2}(t)+t^2-t\sqrt{t^2+\varphi^{2k+2}(t)}}{t^{2k}}\ \frac{1}{(1+\varphi^{2k+2}(t)/t^2)^k}\\
&\xrightarrow[t\to 0^+]{}\ 2\left(\frac{\varphi^{2k+2}(t)}{t^{2k}}+\frac{t-\sqrt{t^2+\varphi^{2k+2}(t)}}{t^{2k-1}}\right)\\
&= 2\left(\frac{\varphi^{2k+2}(t)}{t^{2k}}-\frac{\varphi^{2k+2}(t)}{t^{2k-1}(\sqrt{t^2+\varphi^{2k+2}(t)}+t)}\right)\\
&= 2\left(\frac{\varphi^{2k+2}(t)}{t^{2k}}-\frac{\varphi^{2k+2}(t)}{t^{2k}(\sqrt{1+\varphi^{2k+2}(t)/t^2}+1)}\right)\\
&= 2\frac{\varphi^{2k+2}(t)}{t^{2k}}\left(1-\frac{1}{1+\sqrt{1+\varphi^{2k+2}(t)/t^2}}\right)\xrightarrow[t\to 0^+]{}\left(\frac{\varphi^{k+1}(t)}{t^k}\right)^2\xrightarrow[t\to 0^+]{}0
\end{aligned}$$

what means that

$$\frac{1}{r^k}d_\rho(C\cap S_\rho(p,r))\xrightarrow[r\to 0^+]{}0 \tag{20}$$

From here it follows that for an arbitrary $\varepsilon>0$ there exists $\delta_1>0$ such that

$$\frac{1}{r^k}d_\rho(C\cap S_\rho(p,r))<\frac{\varepsilon}{2}\quad\text{for}\quad 0<r<\delta_1 \tag{21}$$

Now we shall prove that for an arbitrary $\varepsilon>0$ there exists $\delta_2>0$ such that for every pair of points $(x,y_1)\in[A,p;\mu,k]$

$$\frac{\rho(x,y_1)}{\rho^k(p,x)}<\frac{\varepsilon}{2} \tag{22}$$

when

$$r=\rho(p,x)<\delta_2\quad\text{and}\quad\frac{\rho(x,A)}{\rho^k(p,x)}<\delta_2 \tag{23}$$

Let y_1' be a projection of the point $x\in E$ at the set A, i.e., such point of the set A that $\rho(x,y_1')=\rho(x,A)$. Because $x=(t,\pm\sqrt{r^2-t^2})$ for $0\le t<r$, then

$$\rho(y_1', y) = r - t = \sqrt{(r-t)^2} \le \sqrt{(r+t)(r-t)} = \sqrt{r^2 - t^2} = \rho(x, y_1')$$

that is to say,

$$\rho(y_1', y) \le \rho(x, A) \tag{24}$$

Let $\mu = 2$, $\delta_2 = \min(\frac{1}{2}, \frac{\varepsilon}{4})$. Hence, from (23), (24) and from the triangle inequality we have

$$\frac{\rho(x, y_1)}{\rho^k(p, x)} \le \frac{\rho(x, y_1') + \rho(y_1', y)}{\rho^k(p, x)} \le \frac{2\rho(x, A)}{\rho^k(p, x)} < \frac{\varepsilon}{2}$$

which yields the inequality (22).

Lastly we shall prove that for an arbitrary $\varepsilon > 0$ there exists $\delta_3 > 0$ such that for every pair of points $(x, y_2) \in [B, p; \mu, k]$

$$\frac{\rho(x, y_2)}{\rho^k(p, x)} < \frac{\varepsilon}{2} \tag{25}$$

if only

$$r = \rho(p, x) < \delta_3 \quad \text{and} \quad \frac{\rho(x, B)}{\rho^k(p, x)} < \delta_3 \tag{26}$$

From the properties of the function φ it follows that

$$(\varphi^{k+1}(t))'|_{t=0} = 0 \tag{27}$$

what means that the set B is tangent to the axis x at the point p. From here it follows that in a certain right-hand side neighbourhood of 0 the function $y = \varphi^{k+1}(t)$ is a convex function. Let y_2' be a projection of the point $x \in E$ at the set B, i.e., such point of the set B that $\rho(x, y_2') = \rho(x, B)$. Let L be a tangent line to the set B at the point y_2', and let $y \in L \cap S_\rho(p, r)$, where $S_\rho(p, r)$ denotes the sphere with the centre at the point $p \in E$ and the radius $r > 0$ in the metric space (E, ρ). From here, on the base of the inequality (24), it follows that

$$\rho(y_2', y) \le \rho(x, y_2') \le \rho(x, B) \tag{28}$$

Hence and from the triangle inequality we get

$$\rho(x, y_2) \le \rho(x, y) \le \rho(x, y_2') + \rho(y_2', y) \le 2\rho(x, B) \tag{29}$$

Putting $\mu = 2$, $\delta_3 = \min(\frac{1}{2}, \frac{\varepsilon}{4})$, from the inequality (29) we obtain

$$\frac{\rho(x, y_2)}{\rho^k(p, x)} \le \frac{2\rho(x, B)}{\rho^k(p, x)} < \frac{\varepsilon}{2}$$

which yields the inequality (25).

Let $\mu = 2$, $\delta = \min(\delta_1, \delta_2, \delta_3)$ and let (x, y) be an arbitrary pair of points belonging to the set $[C, p; \mu, k]$. In this example: $\rho(x, C) = \rho(x, A)$, or $\rho(x, C) = \rho(x, B)$, or $x \in C$.

Let us suppose that $\rho(x, C) = \rho(x, A)$. From here, from the triangle inequality, from (21) and (22) it follows that for an arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that for every pair of points $(x, y) \in [C, p; \mu, k]$, if

$$r = \rho(p, x) < \delta \quad \text{and} \quad \frac{\rho(x, C)}{\rho^k(p, x)} < \delta$$

then

$$\frac{\rho(x, y)}{\rho^k(p, x)} \le \frac{\rho(x, y_1)}{\rho^k(p, x)} + \frac{\rho(y, y_1)}{\rho^k(p, x)} \le \frac{\rho(x, y_1)}{\rho^k(p, x)} + \frac{1}{r^k} d_\rho(C \cap S_\rho(p, r)) < \varepsilon \tag{30}$$

Similarly, if $\rho(x, C) = \rho(x, B)$ then from here, from the triangle inequality, from (21) and (25) it follows that for an arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that for every pair of points $(x, y) \in [C, p; \mu, k]$, if

$$r = \rho(p, x) < \delta \quad \text{and} \quad \frac{\rho(x, C)}{\rho^k(p, x)} < \delta$$

then

$$\frac{\rho(x, y)}{\rho^k(p, x)} \le \frac{\rho(x, y_2)}{\rho^k(p, x)} + \frac{\rho(y, y_2)}{\rho^k(p, x)} \le \frac{\rho(x, y_2)}{\rho^k(p, x)} + \frac{1}{r^k} d_\rho(C \cap S_\rho(p, r)) < \varepsilon \tag{31}$$

If $x \in C$, then from (21) it follows immediately that for an arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that for every pair of points $(x, y) \in [C, p; \mu, k]$

$$\frac{\rho(x, y)}{\rho^k(p, x)} \le \frac{1}{r^k} d_\rho(C \cap S_\rho(p, r)) < \varepsilon \tag{32}$$

when

$$r = \rho(p, x) < \delta \quad \text{and} \quad \frac{\rho(x, C)}{\rho^k(p, x)} = 0 < \delta$$

Hence, from (30) and (31) it follows that the set C defined by the formula (16) belongs to the class $\widetilde{M_{p,k}}$.

Evidently, the set C of the form (16) has the Darboux property at the point p of the metric space (E, ρ). From the above it follows that $C \in \widetilde{M_{p,k}} \cap D_p(E, \rho)$.

Because the sets A, B defined by the formula (17) have the Darboux property at the point p of the space (E, l), and are subsets of the set $C \in \widetilde{M_{p,k}}$, then from here and from Theorem 2.3 of the paper [7] it follows that the set A is (a, b)-tangent of order k ($k \in \mathbf{N}$) to the set B at the point p of the space (E, l), when $l \in \mathfrak{F}_f$, and the functions a, b fulfil the condition

$$\frac{a(r)}{r^k} \xrightarrow[r \to 0^+]{} 0 \quad \text{and} \quad \frac{b(r)}{r^k} \xrightarrow[r \to 0^+]{} 0 \tag{33}$$

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