THE TANGENCY OF SETS AND MONOTONICITY FUNCTION

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Abstract. The present paper deals with the connections between tangency relations of sets $T(a_i,b_i,k,p)$ ($l = 1,2$) in the metric space $(E,\rho)$ and monotonicity function $l$.

Let $(E,\rho)$ be a metric space. Let the set 

$$S(p,t) = \{x \in E : \rho(p,x) = t\}$$

(1)

denote the sphere with centre of the point $p$ and the radius $t$, and

$$K(p,t) = \{x \in E : \rho(p,x) < t\}$$

(2)

denote the ball with centre of the point $p$ and the radius $t$. The a set 

$$S(p,r) = \bigcup_{q \in S(p,t)} K(q,t) \quad \text{for} \quad t > 0$$

(3)

and

$$S(p,r) = S(p,t) \quad \text{for} \quad t = 0$$

(4)

will be called the $t$ - neighbourhood of the sphere. The $E_0$ be the family of all non-empty subset of set $E$.

Let $a$ and $b$ are non-negative real functions defined in a right-hand neighbourhood of the point 0 such that

$$a(r) \xrightarrow{r \to 0^+} 0 \quad \text{and} \quad a(r) \xrightarrow{r \to 0^-} 0$$

(5)

The pair $(A,B)$ is $(a,b)$ - concentrated at the point $p$ if 0 is a concentration point of the set all real numbers $r > 0$ such that the sets $A \cap S(p,r)_{a(r)}$ and $B \cap S(p,r)_{b(r)}$ are non-empty.

Let $l$ be a real non-negative function defined on the Cartesian product $E_0 \times E_0$ satisfying the condition

$$l(\{x\},\{y\}) = \rho(x,y) \quad \text{for} \quad x, y \in E$$

(6)
The set $A$ is $(a,b)$ - tangent order $k$ at the point $p \in E$ to the set $B$ if the pair $(A,B)$ is $(a,b)$ - concentrated at the point $p$ and

$$\frac{1}{r^k} I(A \cap S(p,r)_{a(r)}, B \cap S(p,r)_{b(r)}) \xrightarrow{r \to 0^+} 0$$

(7)

where $k$ is an arbitrary positive real number. The relation

$$T_i(a,b,k,p) = \left\{ (A,B) \in E_0 \land \frac{1}{r^k} I(A \cap S(p,r)_{a(r)}, B \cap S(p,r)_{b(r)}) \xrightarrow{r \to 0^+} 0 \right\}$$

(8)

will be called the relations of tangency of the set in the metric space $(E,\rho)$.

**THEOREM 9.** If $0 \leq t_1 \leq t_2$, $t_1, t_2 \in R$ (the $R$ is set of real numbers), then

$$S(p,r)_{t_1} \subset S(p,r)_{t_2}$$

Proof. Let $t_1 > 0$ then and $x \in S(p,r)_{t_1}$ from (3) exists a point $q \in S(p,r)$ such that $x \in K(q,t_1)$, there $\rho(q,x) < t_1 \leq t_2$ and so $x \in K(p,t_2)$ to say $x \in S(q,t)_{t_2}$. Let $t_1 = 0$ then $S(p,r)_{t_1} = S(p,r)$. Let $x \in S(p,r)$ to say $\rho(p,x) = r$ and $x \in K(p,t_2)$, therefore $x \in S(q,t)_{t_2}$. This ends the proof.

**THEOREM 10.** If

$$l(A,B) \leq l(C,B) \quad \text{for } A \subset C \quad (A,B,C \in E_0)$$

(11)

and $a_i(r) \leq a_j(r)$ for $r > 0$, then

$$(A,B) \in T_i(a_1,b,k,p) \Rightarrow (A,B) \in T_i(a_j,b,k,p)$$

Proof. Let $(A,B) \in T_i(a_1,b,k,p)$ and $a_i(r) \leq a_j(r)$ for $r > 0$. Then $a_i(r) \xrightarrow{r \to 0^+} 0$ from here and from theorem (9) $S(p,r)_{a_i(r)} \subset S(p,r)_{a_j(r)}$ for say

$$\frac{1}{r^k} I(A \cap S(p,r)_{a_j(r)}, B \cap S(p,r)_{a_j(r)}) \leq \frac{1}{r^k} I(A \cap S(p,r)_{a_i(r)}, B \cap S(p,r)_{a_i(r)})$$

from here $(A,B) \in T_i(a_1,b,k,p)$. This ends the proof.
Example 12. The function
\[ l_s(A, B) = \sup \{\text{diam}(\{x\} \cup B) ; x \in A_1^{\prime}\} \]

by \textit{diam}A we shall denote the diameter of the set \( A \), the \( A, B \in E_0 \), satisfying the condition (6) and (11), therefore for
\[ (A, B) \in T_1(a_1, b, k, p) \Rightarrow (A, B) \in T_1(a_2, b, k, p) \]

THEOREM 13. If
\[ l(A, B) \leq l(A, D) \text{ for } B \subset D \quad (A, B, D \in E_0) \tag{14} \]

and \( b_1(r) \leq b_2(r) \) for \( r > 0 \), then
\[ (A, B) \in T_1(a, b_1, k, p) \subseteq (A, B) \in T_1(a, b_2, k, p) \]

Proof. Let \( (A, B) \in T_1(a, b_2, k, p) \) and \( b_1(r) \leq b_2(r) \) for \( r > 0 \). Then \( b_1(r) \xrightarrow{r \to 0^+} 0 \) from here and from theorem (9) \( S(p, r)_{b_1(r)} \subset S(p, r)_{b_2(r)} \) for say
\[ \frac{1}{r^2} l(A \cap S(p, r)_{a_1(r)}, B \cap S(p, r)_{b_1(r)}) \leq \frac{1}{r^2} l(A \cap S(p, r)_{a_2(r)}, B \cap S(p, r)_{b_2(r)}) \]

from here \( (A, B) \in T_1(a, b_2, k, p) \). This ends the proof.

Example 15. The function \( l_t \) by the formula
\[ l_t(A, B) = \text{diam}(A \cup B) \]

satisfying the condition (6) and (14), therefore for \( a_1(r) \leq a_2(r) \) as \( r > 0 \)
\[ (A, B) \in T_1(a_1, b_1, k, p) \subseteq (A, B) \in T_1(a_2, b_2, k, p) \]

THEOREM 16. If
\[ l(A, B) \geq l(C, B) \text{ for } A \subset C \quad (A, B, C \in E_0) \tag{17} \]

and \( a_1(r) \leq a_2(r) \) for \( r > 0 \) and \( a_2(r) \xrightarrow{r \to 0^+} 0 \), then
\[ (A, B) \in T_1(a_2, b, k, p) \Rightarrow (A, B) \in T_1(a_1, b, k, p) \]
Proof. Let \((A, B) \in T_1(a_2, b, k, p)\) and \(a_i(r) \leq a_2(r)\) for \(r > 0\). Then \(a_i(r) \rightarrow 0\) as \(r \rightarrow 0^+\) from here and from theorem (9) \(S(p, r)_{a_1(r)} \subset S(p, r)_{a_2(r)}\) for say

\[
\frac{1}{r^k} l\left(\left(A \cap S(p, r)_{a_1(r)}\right) \cap S(p, r)_{a_2(r)}\right) \leq \frac{1}{r^k} l\left(\left(A \cap S(p, r)_{a_1(r)}\right) \cap S(p, r)_{a_2(r)}\right)
\]

from here \((A, B) \in T_1(a_2, b, k, p)\). This ends the proof.

Example 18. The function \(l_6\) by the formula

\[
l_6(A, B) = \inf\{\text{diam}\{(x) \cup B\}; x \in A\}
\]

satisfying the condition (6) and (14), therefore for \(a_i(r) \leq a_2(r)\) as \(r > 0\)

\[(A, B) \in T_1(a, b_1, k, p) \Rightarrow (A, B) \in T_1(a, b_2, k, p)\]

THEOREM 19. If

\[
l(A, B) \geq l(A, D) \quad \text{for} \quad B \subset D \quad (A, B, D \in E_0)
\]

and \(b_1(r) \leq b_2(r)\) for \(r > 0\) and \(b_2(r) \rightarrow 0^+\) as \(r \rightarrow 0^+\), then

\[(A, B) \in T_1(a, b_1, k, p) \Rightarrow (A, B) \in T_1(a, b_2, k, p)\]

Proof. Let \((A, B) \in T_1(a, b_1, k, p)\) and \(a_i(r) \leq a_2(r)\) for \(r > 0\). Then from here and from theorem (9) \(S(p, r)_{a_1(r)} \subset S(p, r)_{a_2(r)}\) for say

\[
\frac{1}{r^k} l\left(\left(A \cap S(p, r)_{a_1(r)}\right) \cap S(p, r)_{a_2(r)}\right) \leq \frac{1}{r^k} l\left(\left(A \cap S(p, r)_{a_1(r)}\right) \cap S(p, r)_{a_2(r)}\right)
\]

from here \((A, B) \in T_1(a_2, b, k, p)\). This ends the proof.

The metric \(\rho\) induces some \(l_i(x)\) defined by formulas:

\[
\begin{align*}
l_i(A, B) &= \inf\{\inf\{\rho(x, y); y \in B\}; x \in A\} \\
l_2(A, B) &= \inf\{\sup\{\rho(x, y); y \in B\}; x \in A\} \\
l_3(A, B) &= \inf\{\sup\{\rho(x, y); y \in B\}; x \in A\} \\
l_4(A, B) &= \sup\{\inf\{\rho(x, y); y \in B\}; x \in A\} \\
l_5(A, B) &= \sup\{\sup\{\rho(x, y); y \in B\}; x \in A\} \\
l_6(A, B) &= \inf\{\text{diam}\{(x) \cup B\}; x \in A\} \\
l_7(A, B) &= \inf\{\text{diam}\{(x) \cup B\}; x \in A\} \\
l_8(A, B) &= \text{diam}\{(x) \cup B\}; x \in A\}
\end{align*}
\]
The function $l_1$ satisfying conditions (17) and (19). The function $l_2$ satisfying condition (11) and (14). The function $l_3$ satisfying condition (17) and (14). The function $l_4$ satisfying condition (11) and (14).

**THEOREM 21.** For $A, B ∈ E_0$

$$l_4(A, B) = \max\{\text{diam}A, \text{diam}B, l_4(A, B)\}$$

Proof. Let $\text{diam}(A ∪ B) = s < ∞$. Let the $ε > 0$, exist point $x, y ∈ A ∪ B$ such as this $s − ε < ρ(x, y) ≤ s$. Then at least one inequality:

- $s − ε < ρ(x, y) ≤ s$ for $x, y ∈ A$
- $s − ε < ρ(x, y) ≤ s$ for $x, y ∈ B$
- $s − ε < ρ(x, y) ≤ s$ for $x ∈ A, y ∈ B$

is true.

From here $\text{diam}(A ∪ B) = \text{diam}A$ and $\text{diam}B ≤ \text{diam}A$ and $l_4(A, B) ≤ \text{diam}A$

or $\text{diam}(A ∪ B) = \text{diam}B$ and $\text{diam}A ≤ \text{diam}B$ and $l_4(A, B) ≤ \text{diam}B$

else $\text{diam}(A ∪ B) = l_4(A, B)$ and $\text{diam}A ≤ l_4(A, B)$ and $\text{diam}B ≤ l_4(A, B)$

therefore

$$\text{diam}(A ∪ B) = \max\{\text{diam}A, \text{diam}B, l_4(A, B)\}$$

Let $\text{diam}(A ∪ B) = ∞$, then exist point $x, y ∈ A ∪ B$ that such $ρ(x, y) > N$, where $N$ is arbitrary positive real number, then at least one inequality: $ρ(x, y) > N$ for $x, y ∈ A$, $ρ(x, y) > N$ for $x, y ∈ B$, $ρ(x, y) > N$ for $x ∈ A, y ∈ B$. From here $\text{diam}A = ∞$ or $\text{diam}B = ∞$ else $l_4(A, B) = ∞$. This ends the proof.

**THEOREM 22.** For $A, B ∈ E_0$

$$l_5(A, B) = \max\{\text{diam}(A, l_3(A, B))\}$$

Proof. Let the $A = \{x\}$ for a $x ∈ E$ to say $\text{diam}A = 0$. Then from Theorem (21)

$$\text{diam}(\{x\} ∪ B) = \max\{\text{diam}B, \sup\{ρ(x, y) ; y ∈ B\}\}$$

Therefore

$$l_5(A, B) = \inf\{\max\{\text{diam}B, \sup\{ρ(x, y) ; y ∈ B\}\} ; x ∈ A\}$$

Let

$$\max\{\text{diam}B, \sup\{ρ(x, y) ; y ∈ B\} ; x ∈ A\} = \text{diam}B$$

that is

$$\inf\{\sup\{ρ(x, y) ; y ∈ B\} ; x ∈ A\} ≤ \text{diam}B$$
to say exist a point \( x_0 \in A \) such that \( \sup\{\rho(x_0, y); y \in B\} \leq \text{diam}B \). For an arbitrary point \( x \in A \)

\[
\text{diam}B \leq \inf\{\max\{\text{diam}B, \sup\{\rho(x, y); y \in B\}; x \in A\}\} \leq \\
\leq \max\{\text{diam}B, \sup\{\rho(x_0, y); y \in B\}\} = \text{diam}B
\]

Let

\[
\max\{\text{diam}B, \inf\{\sup\{\rho(x, y); y \in B\}; x \in A\}\} = \inf\{\sup\{\rho(x, y); y \in B\}; x \in A\}
\]

then

\[
\inf \geq \{\sup\{\rho(x, y); y \in B\}; x \in A\} \geq \text{diam}B
\]

that is

\[
\sup\{\rho(x, y); y \in B\} \geq \text{diam}B \quad \text{for} \quad x \in A
\]

For arbitrary point \( x \in A \)

\[
\sup\{\rho(x, y); y \in B\} \leq \max\{\text{diam}B, \sup\{\rho(x, y); y \in B\}\}
\]

from here

\[
\inf\{\sup\{\rho(x, y); y \in B\}; x \in A\} = \inf\{\max\{\text{diam}B, \sup\{\rho(x, y); y \in B\}\}; x \in A\} \leq \\
\leq \max\{\text{diam}B, \sup\{\rho(x, y); y \in B\}\} \leq \sup\{\rho(x, y); y \in B\}
\]

Therefore

\[
\inf\{\sup\{\rho(x, y); y \in B\}; x \in A\} = \inf\{\max\{\text{diam}B, \sup\{\rho(x, y); y \in B\}\}; x \in A\}
\]

This ends the proof.

References