THE TANGENCY OF SETS AND MONOTONICITY FUNCTION

Jerzy Grochulski

Institute of Mathematics and Computer Science, Czestochowa University of Technology

Abstract. The present paper deals with the connections between tangency relations of sets $T_l(a_i,b_i,k,p)$ ($l = 1,2$) in the metric space (E,ρ) and monotonicity function l.

Let (E,ρ) be a metric space. Let the set

$$S(p,t)=\{x\in E:\rho(p,x)=t\} \tag{1}$$

denote the sphere with centre of the point p and the radius t, and

$$K(p,t)=\{x\in E:\rho(p,x)<t\} \tag{2}$$

denote the ball with centre of the point p and the radius t. The a set

$$S(p,r)_t = \bigcup_{q\in S(p,r)} K(q,t) \quad \text{for} \quad t>0 \tag{3}$$

and

$$S(p,r)_t = S(p,t) \quad \text{for} \quad t=0 \tag{4}$$

will be called the t - neighbourhood of the sphere. The E_0 be the family of all non--empty subset of set E.

Let a and b are non-negative real functions defined in a right-hand neighbourhood of the point 0 such that

$$a(r)\xrightarrow[r\to 0^+]{}0 \quad \text{and} \quad a(r)\xrightarrow[r\to 0^+]{}0 \tag{5}$$

The pair (A,B) is (a,b) - concentrated at the point p if 0 is a concentration point of the set all real numbers $r > 0$ such that the sets $A\cap S(p,r)_{a(r)}$ and $B\cap S(p,r)_{b(r)}$ are non-empty.

Let l be a real non-negative function defined on the Cartesian product E_0xE_0 satisfying the condition

$$l(\{x\},\{y\})=\rho(x,y) \quad \text{for} \quad x,\, y\in E \tag{6}$$

The set A is (a,b) - tangent order k at the point $p \in E$ to the set B if the pair (A,B) is (a,b) - concentrated at the point p and

$$\frac{1}{r^k} l\left(A \cap S(p,r)_{a(r)}, B \cap S(p,r)_{b(r)}\right) \xrightarrow[r \to 0^+]{} 0 \tag{7}$$

where k is an arbitrary positive real number. The relation

$$T_l(a,b,k,p) = \\ = \left\{ (A,B) : A,B \in E_0 \wedge \frac{1}{r^k} l\left(A \cap S(p,r)_{a(r)}, B \cap S(p,r)_{b(r)}\right) \xrightarrow[r \to 0^+]{} 0 \right\} \tag{8}$$

will be called the relations of tangency of the set in the metric space (E,ρ).

THEOREM 9. If $0 \le t_1 \le t_2$, $t_1, t_2 \in R$ (the R is set of real numbers), then

$$S(p,r)_{t_1} \subset S(p,r)_{t_2}$$

Proof. Let $t_1 > 0$ then and $x \in S(p,r)_{t_1}$ from (3) exists a point $q \in S(p,r)$ such that $x \in K(q,t_1)$, there $\rho(q,x) < t_1 \le t_2$ and so $x \in K(p,t_2)$ to say $x \in S(q,t)_{t_2}$. Let $t_1 = 0$ then $S(p,r)_{t_1} = S(p,r)$. Let $x \in S(p,r)$ to say $\rho(p,x) = r$ and $x \in K(p,t_2)$, therefore $x \in S(q,t)_{t_2}$. This ends the proof.

THEOREM 10. If

$$l(A,B) \le l(C,B) \quad \text{for } A \subset C \quad (\mathrm{A,B,C} \in \mathrm{E}_0) \tag{11}$$

and $a_1(r) \le a_2(r)$ for $r > 0$, then

$$(A,B) \in T_1(a_2,b,k,p) \Rightarrow (A,B) \in T_1(a_1,b,k,p)$$

Proof. Let $(A,B) \in T_1(a_2,b,k,p)$ and $a_1(r) \le a_2(r)$ for $r > 0$. Then $a_1(r) \xrightarrow[r \to 0^+]{} 0$ from here and from theorem (9) $S(p,r)_{a_1(r)} \subset S(p,r)_{a_2(r)}$ for say

$$\frac{1}{r^k} l\left(A \cap S(p,r)_{a_1(r)}, B \cap S(p,r)_{a_2(r)}\right) \le \frac{1}{r^k} l\left(A \cap S(p,r)_{a_2(r)}, B \cap S(p,r)_{b(r)}\right)$$

from here $(A,B) \in T_1(a_1,b,k,p)$. This ends the proof.

Example 12. The function

$$l_5(A,B)=\sup\{diam(\{x\}\cup B);x\in A\}$$

by *diamA* we shall denote the diameter of the set *A*, the $A,B\in E_0$, satisfying the condition (6) and (11), therefore for

$$(A,B)\in T_1(a_1,b,k,p)\Rightarrow(A,B)\in T_1(a_2,b,k,p)$$

THEOREM 13. If

$$l(A,B)\le l(A,D)\quad \text{for } B\subset D\quad (A,B,D\in E_0) \tag{14}$$

and $b_1(r)\le b_2(r)$ for $r>0$, then

$$(A,B)\in T_1(a,b_1,k,p)\Leftarrow(A,B)\in T_1(a,b_2,k,p)$$

Proof. Let $(A,B)\in T_1(a,b_2,k,p)$ and $b_1(r)\le b_2(r)$ for $r>0$. Then $b_1(r)\xrightarrow[r\to 0^+]{}0$ from here and from theorem (9) $S(p,r)_{b_1(r)}\subset S(p,r)_{b_2(r)}$ for say

$$\frac{1}{r^k}l\left(A\cap S(p,r)_{a(r)},B\cap S(p,r)_{b_1(r)}\right)\le\frac{1}{r^k}l\left(A\cap S(p,r)_{a(r)},B\cap S(p,r)_{b_2(r)}\right)$$

from here $(A,B)\in T_1(a,b_2,k,p)$. This ends the proof.

Example 15. The function l_7 by the formula

$$l_7(A,B)=diam(A\cup B)$$

satisfying the condition (6) and (14), therefore for $a_1(r)\le a_2(r)$ as $r>0$

$$(A,B)\in T_1(a,b_1,k,p)\Leftarrow(A,B)\in T_1(a,b_2,k,p)$$

THEOREM 16. If

$$l(A,B)\ge l(C,B)\quad \text{for } A\subset C\quad (A,B,C\in E_0) \tag{17}$$

and $a_1(r)\le a_2(r)$ for $r>0$ and $a_2(r)\xrightarrow[r\to 0^+]{}0$, then

$$(A,B)\in T_1(a_2,b,k,p)\Rightarrow(A,B)\in T_1(a_1,b,k,p)$$

Proof. Let $(A,B)\in T_1(a_2,b,k,p)$ and $a_1(r)\le a_2(r)$ for $r > 0$. Then $a_1(r)\xrightarrow[r\to 0^+]{}0$ from here and from theorem (9) $S(p,r)_{a_1(r)}\subset S(p,r)_{a_2(r)}$ for say

$$\frac{1}{r^k}l\left(A\cap S(p,r)_{a_1(r)},B\cap S(p,r)_{a_2(r)}\right)\le\frac{1}{r^k}l\left(A\cap S(p,r)_{a_1(r)},B\cap S(p,r)_{b(r)}\right)$$

from here $(A,B)\in T_1(a_2,b,k,p)$. This ends the proof.

Example 18. The function l_6 by the formula

$$l_6(A,B)=\inf\{diam(\{x\}\cup B);x\in A\}$$

satisfying the condition (6) and (14), therefore for $a_1(r)\le a_2(r)$ as $r > 0$

$$(A,B)\in T_1(a,b_1,k,p)\Rightarrow(A,B)\in T_1(a,b_2,k,p)$$

THEOREM 19. If

$$l(A,B)\ge l(A,D)\quad \text{for } B\subset D\quad (A,B,D\in E_0) \tag{20}$$

and $b_1(r)\le b_2(r)$ for $r > 0$ and $b_2(r)\xrightarrow[r\to 0^+]{}0$, then

$$(A,B)\in T_1(a,b_1,k,p)\Rightarrow(A,B)\in T_1(a,b_2,k,p)$$

Proof. Let $(A,B)\in T_1(a,b_1,k,p)$ and $a_1(r)\le a_2(r)$ for $r > 0$. Then from here and from theorem (9) $S(p,r)_{a_1(r)}\subset S(p,r)_{a_2(r)}$ for say

$$\frac{1}{r^k}l\left(A\cap S(p,r)_{a(r)},B\cap S(p,r)_{b_2(r)}\right)\le\frac{1}{r^k}l\left(A\cap S(p,r)_{a(r)},B\cap S(p,r)_{b_1(r)}\right)$$

from here $(A,B)\in T_1(a_2,b,k,p)$. This ends the proof.

The metric ρ induces some l_i(x) defined by formulas:

$$l_1(A,B)=\inf\{\inf\{\rho(x,y);y\in B\};x\in A\}$$
$$l_2(A,B)=\sup\{\inf\{\rho(x,y);y\in B\};x\in A\}$$
$$l_3(A,B)=\inf\{\sup\{\rho(x,y);y\in B\};x\in A\}$$
$$l_4(A,B)=\sup\{\sup\{\rho(x,y);y\in B\};x\in A\}$$
$$l_5(A,B)=\sup\{diam\{x\}\cup B;x\in A\}$$
$$l_6(A,B)=\inf\{diam(\{x\}\cup B);x\in A\}$$
$$l_7(A,B)=diam(A\cup B)$$

The function l_1 satisfying conditions (17) and (19). The function l_2 satisfying condition (11) and (14). The function l_3 satisfying condition (17) and (14). The function l_4 satisfying condition (11) and (14).

THEOREM 21. For $A, B \in E_0$

$$l_8(A,B) = \max\{diamA, diamB, l_4(A,B)\}$$

Proof. Let $diam(A \cup B) = s < \infty$. Let the $\varepsilon > 0$, exist point $x, y \in A \cup B$ such as this $s - \varepsilon < \rho(x,y) \le s$. Then at least one inequality:

$$s - \varepsilon < \rho(x,y) \le s \quad \text{for } x, y \in A$$
$$s - \varepsilon < \rho(x,y) \le s \quad \text{for } x, y \in B$$
$$s - \varepsilon < \rho(x,y) \le s \quad \text{for } x \in A, y \in B$$

is true.

From here $diam(A \cup B) = diamA$ and $diamB \le diamA$ and $l_4(A,B) \le diamA$
or $diam(A \cup B) = diamB$ and $diamA \le diamB$ and $l_4(A,B) \le diamB$
else $diam(A \cup B) = l_4(A,B)$ and $diamA \le l_4(A,B)$ and $diamB \le l_4(A,B)$
therefore

$$diam(A \cup B) = \max\{diamA, diamB, l_4(A < B)\}$$

Let $diam(A \cup B) = \infty$, then exist point $x, y \in A \cup B$ that such $\rho(x,y) > N$, where N is arbitrary positive real number, then at least one inequality: $\rho(x,y) > N$ for $x, y \in A$, $\rho(x,y) > N$ for $x, y \in B$, $\rho(x,y) > N$ for $x \in A, y \in B$. From here $diamA = \infty$ or $diamB = \infty$ else $l_4(A,B) = \infty$. This ends the proof.

THEOREM 22. For $A, B \in E_0$ $l_9(A,B) = \max\{diam(A, l_3(A,B))\}$.
Proof. Let the $A = \{x\}$ for a $x \in E$ to say $diamA = 0$. Then from Theorem (21)

$$diam(\{x\} \cup B) = \max\{diamB, \sup\{\rho(x,y); y \in B\}\}$$

Therefore

$$l_7(A,B) = \inf\{\max\{diamB, \sup\{\rho(x,y); y \in B\}\}; x \in A\}$$

Let

$$\max\{diamB, \inf\{\sup\{\rho(x,y); y \in B\}; x \in A\}\} = diamB$$

that is

$$\inf\{\sup\{\rho(x,y); y \in B\}; x \in\} \le diamB$$

to say exist a point $x_0 \in A$ such that $\sup\{\rho(x_0, y); y \in B\} \le diamB$. For an arbitrary point $x \in A$

$$diamB \le \inf\{\max\{diamB, \sup\{\rho(x, y); y \notin B\}\}; x \in A\} \le$$
$$\le \max\{diamB, \sup\{\rho(x_0, y); y \in B\}\} = diamB$$

Let

$$\max\{diamB, \inf\{\sup\{\rho(x, y); y \in B\}; x \in A\}\} = \inf\{\sup\{\rho(x, y); y \in B\}; x \in A\}$$

then

$$\inf \ge \{\sup\{\rho(x, y); y \in B\}; x \in A\} \ge diamB$$

that is

$$\sup\{\rho(x, y); y \in B\} \ge diamB \quad \text{for } x \in A$$

For arbitrary point $x \in A$

$$\sup\{\rho(x, y); y \in B\} \le \max\{diamB, \sup\{\rho(x, y); y \in B\}\}$$

from here

$$\inf\{\sup\{\rho(x, y); y \in B\}; x \in A\} = \inf\{\max\{diamB, \sup\{\rho(x, y); y \in B\}\}; x \in A\} \le$$
$$\le \max\{diamB, \sup\{\rho(x, y); y \in B\}\} \le \sup\{\rho(x, y); y \in B\}$$

Therefore

$$\inf\{\sup\{\rho(x, y); y \in B\}; \in A\} = \inf\{\max\{diamB, \sup\{\rho(x, y); y \in B\}\}; x \in A\}$$

This ends the proof.

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