THE INFINITY PHENOMENON FOR POLYNOMIAL MAPPING

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Abstract. In the present paper I give the formula for the residue at infinity of polynomial mapping with *n*-variables in the case when the numerator degree in the formula defining residuum is not smaller than the sum of degrees of mapping components - *n*.

1. Introduction

Made repeatedly by me an attempt to make a definition of not integral residue at infinity of polynomial mapping with *n*-variables has faced essential difficulties. Previous, fragmentary results included only the case when the numerator degree in the formula defining residue is smaller than the sum of degrees of mapping components - *n* (see [1]). In presented paper I give the formula for the residue at the infinity of polynomial mapping also in the case when the numerator degree remains not smaller than the sum of degrees of mapping components - *n* (see point 3. Definition of the residue at infinity). The correct formulating of this definition enables the lemma formulated in the point 2. It is what characterizes unexpected before behaviour of polynomial mapping at infinity emphasised it the title of the article. Parameterization formulas following the definition are just the conclusion from the previously obtained by me the formula for parametric residue (see [2]).

2. Basic lemma

Let us consider the polynomial mapping $F=(f_1,...,f_n)$ with complex *n*-variables with isolated zeros at infinity. Hyperplane at infinity $H_\infty = V(T_0)$ in the complex projective space $\boldsymbol{CP}^n$, with homogeneous coordinates $T_0, T_1, ..., T_n$, we chose in that way, that all from the algebraic sets being intersection of the closer in $\boldsymbol{CP}^n$ of the set of zeros $V(f_1,...,f_{j-1},f_{j+1},...,f_n)$ of polynomials $f_1,...,f_{j-1},f_{j+1},...,f_n$ with H_∞ was finite. Let further V_j be the closer in $\boldsymbol{CP}^n$ of the set $V(f_1,...,f_{j-1},f_{j+1},...,f_n)$. Moreover let us denote by $\widetilde{f}_1,...,\widetilde{f}_n$ canonical images of polynomials $f_1,...,f_n$ in the affine part $\boldsymbol{CP}^n - V(T_1) \cong \boldsymbol{C}^n$. Therefore we have $\widetilde{f}_j(X_1,X_2,...,X_n) = X_1^{\deg f_j} f_j(1/X_1, X_2/X_1,...,X_n/X_1)$ for $1 \le j \le n$.

Lemma. *Let h be a complex n-variables polynomial with degree* $\deg h \geq \deg f_1 + \ldots + \deg f_n - n$. *Let* $s = \deg h - (\deg f_1 + \ldots + \deg f_n) + n + 1$.
Equality holds

$$\sum_{a \in V_k \cap H_\infty} \operatorname{Res}_{\tilde{a}} h \big/ \left(\tilde{f}_1, \ldots, X_1^s \tilde{f}_k, \ldots, \tilde{f}_n\right) = \sum_{a \in V_l \cap H_\infty} \operatorname{Res}_{\tilde{b}} h \big/ \left(\tilde{f}_1, \ldots, X_1^s \tilde{f}_l, \ldots, \tilde{f}_n\right)$$

where $\tilde{a}, \tilde{b}$ *are canonical images of points a, b in the affine part* $\boldsymbol{CP}^n - V(T_1) \cong \boldsymbol{C}^n$.

Proof. This is exactly this phenomenon. The sum of residue does not react to the sets V_j. We set $1 \leq j \leq n$ and let us consider mapping $\tilde{F}_j = \left(\tilde{f}_1, \ldots, \tilde{f}_j \tilde{f}_{j+1}, \tilde{f}_{j+2}, \ldots, \tilde{f}_n, X_1^s\right)$. Let point $c \in (V_j \cap V_{j+1}) \cap H_\infty$ and let sums $S_1 \cup \ldots \cup S_p$ and $T_1 \cup \ldots \cup T_q$ denote decomposition of germs of analytic sets $V\left(\tilde{f}_1, \ldots, \hat{\tilde{f}}_j, \ldots, \tilde{f}_n\right)$ and $V\left(\tilde{f}_1, \ldots, \hat{\tilde{f}}_{j+1}, \ldots, \tilde{f}_n\right)$ into irreducible components in the neighbourhood of point $\tilde{c}$ ($\hat{\tilde{f}}_k$ signify omission of coordinate $\tilde{f}_k$). Now, if we take the parametrization $\Phi_k = (\varphi_{k1}, \ldots, \varphi_{kn})$ and $\Psi_l = (\psi_{l1}, \ldots, \psi_{ln})$ respectively of the components S_k and T_l and apply the parametrization theorem of the residue (see [2]), than we will obtain the equalities

$$\operatorname{Res}_{\tilde{c}} h \big/ \tilde{F}_j = \sum_{1 \leq k \leq p} \operatorname{res}_0 \frac{(h \circ \Phi_k)\ \left(\varphi_{k1}^s\right)'}{\left(\operatorname{Jac} \tilde{F}_j \circ \Phi_k\right)\ \varphi_{k1}^s} + \sum_{1 \leq l \leq q} \operatorname{res}_0 \frac{(h \circ \Psi_k)\ \left(\psi_{l1}^s\right)'}{\left(\operatorname{Jac} \tilde{F}_j \circ \Psi_k\right)\ \varphi_{l1}^s} =$$

$$\sum_{1 \leq k \leq p} \operatorname{res}_0 \frac{(h \circ \Phi_k)\ \left(\left(\tilde{f}_j \circ \Phi_k\right)\ \varphi_{k1}^s\right)'}{\left(\operatorname{Jac}\left(\tilde{f}_1, \ldots, \hat{\tilde{f}}_j, \ldots, \tilde{f}_n, \tilde{f}_j X_1^s\right) \circ \Phi_k\right)\ \left(\tilde{f}_j \circ \Phi_k\right)\ \varphi_{k1}^s} +$$

$$\sum_{1 \leq l \leq q} \operatorname{res}_0 \frac{(h \circ \Psi_l)\ \left(\left(\tilde{f}_{j+1} \circ \Psi_l\right)\ \psi_{l1}^s\right)'}{\left(\operatorname{Jac}\left(\tilde{f}_1, \ldots, \hat{\tilde{f}}_{j+1}, \ldots, \tilde{f}_n, \tilde{f}_{j+1} X_1^s\right) \circ \Psi_l\right)\ \left(\tilde{f}_{j+1} \circ \Psi_l\right)\ \psi_{l1}^s} =$$

$$\operatorname{Res}_{\tilde{c}} h \Big/ \left(\tilde{f}_1, \ldots, \hat{\tilde{f}}_j, \ldots, \tilde{f}_n, \tilde{f}_j X_1^s\right) + \operatorname{Res}_{\tilde{c}} h \Big/ \left(\tilde{f}_1, \ldots, \hat{\tilde{f}}_{j+1}, \ldots, \tilde{f}_n, \tilde{f}_{j+1} X_1^s\right) =$$

$$(-1)^{n-j} \left(\operatorname{Res}_{\tilde{c}} h \big/ \left(\tilde{f}_1, \ldots, \tilde{f}_j X_1^s, \ldots, \tilde{f}_n\right) - \operatorname{Res}_{\tilde{c}} h \big/ \left(\tilde{f}_1, \ldots, \tilde{f}_{j+1} X_1^s, \ldots, \tilde{f}_n\right)\right)$$

Thus

$$(-1)^{n-j} \operatorname{Res}_{\tilde{c}} h \big/ \tilde{F}_j = \operatorname{Res}_{\tilde{c}} h \big/ \left(\tilde{f}_1, \ldots, \tilde{f}_j X_1^s, \ldots, \tilde{f}_n\right) - \operatorname{Res}_{\tilde{c}} h \big/ \left(\tilde{f}_1, \ldots, \tilde{f}_{j+1} X_1^s, \ldots, \tilde{f}_n\right)$$

If point $a \in V_j \cap H_\infty$ and $\tilde{f}_j(\tilde{a}) \neq 0$, then as a rule of residue transformation we obtain

$$(-1)^{n-j} \operatorname{Res}_{\tilde{a}} h/\tilde{F}_j = \operatorname{Res}_{\tilde{a}} h/\left(\tilde{f}_1,...,\tilde{f}_j X_1^s,...,\tilde{f}_n\right)$$

and if point $b \in V_{j+1} \cap H_\infty$ and $\tilde{f}_j(\tilde{a}) \neq 0$, then similarly

$$(-1)^{n-j} \operatorname{Res}_{\tilde{b}} h/\tilde{F}_j = \operatorname{Res}_{\tilde{b}} h/\left(\tilde{f}_1,...,\tilde{f}_{j+1} X_1^s,...,\tilde{f}_n\right)$$

Summarising above three equations we obtain following

$$(-1)^{n-j} \sum\nolimits_{\tilde{c} \in \tilde{F}_j^{-1}(0)} \operatorname{Res}_{\tilde{c}} h/\tilde{F}_j =$$
$$\sum\nolimits_{a \in V_j \cap H_\infty} \operatorname{Res}_{\tilde{a}} h/\left(\tilde{f}_1,...,X_1^s \tilde{f}_j,...,\tilde{f}_n\right) - \sum\nolimits_{b \in V_{j+1} \cap H_\infty} \operatorname{Res}_{\tilde{b}} h/\left(\tilde{f}_1,...,X_1^s \tilde{f}_{j+1},...,\tilde{f}_n\right)$$

Because mapping $\tilde{F}_j$ does not have zeros in the infinity as well as degree $\deg h > \deg \tilde{f}_1 + ... + \deg \tilde{f}_n + s - n,$ therefore from the Euler-Jacobi formula (see [3, 4]) it results that the sum on the left side of the above equality is equal to zero. It gives

$$\sum\nolimits_{a \in V_j \cap H_\infty} \operatorname{Res}_{\tilde{a}} h/\left(\tilde{f}_1,...,X_1^s \tilde{f}_j,...,\tilde{f}_n\right) = \sum\nolimits_{b \in V_{j+1} \cap H_\infty} \operatorname{Res}_{\tilde{b}} h/\left(\tilde{f}_1,...,X_1^s \tilde{f}_{j+1},...,\tilde{f}_n\right)$$

and because $1 \leq j \leq n$ was arbitrary chosen, therefore it also give the equality announced by lemma. This ends the proof.

3. Definition of the residue at infinity

Basic lemma enables to define the residue of polynomial mapping at infinity in the case when the numerator degree remains not smaller than the sum of degrees of mapping components - n.

And namely, for a pair consisting of polynomial h and mapping $F = (f_1,...,f_n)$ we assume $\sigma = \deg f_1 + ... + \deg f_n - \deg h - n - 1$ and define residue of this pair at infinity by the equalities

$$\operatorname{Res}_\infty h/F = \sum\nolimits_{a_1 \in V_1 \cap H_\infty} \operatorname{Res}_{\tilde{a}_1} \tilde{h}/\left(X_1^{-\sigma} \tilde{f}_1,...,\tilde{f}_n\right) =$$
$$... = \sum\nolimits_{a_n \in V_n \cap H_\infty} \operatorname{Res}_{\tilde{a}_n} \tilde{h}/\left(\tilde{f}_1,...,X_1^{-\sigma} \tilde{f}_n\right) \text{ for } \sigma < 0$$

4. Parametrization formulas

Let point c be an isolated zero at infinity of polynomial mapping $F=(f_1,\ldots,f_{n-1},f_n)$. Let us denote by $S_1\cup\ldots\cup S_r$ decomposition of the germ of analytic set $V(\tilde{f}_1,\ldots,\tilde{f}_{n-1})$ into irreducible components in the neighbourhood of the point $\tilde{c}$ and we assume that $S_i\cap V(\operatorname{Jac}\tilde{F})\subset\{0\}$ for all $1\le i\le r$. Let $\Phi_i(t)=(t^{\mu_i},\varphi_2^i(t),\ldots,\varphi_n^i(t))$ be a parametrization of components S_i, $1\le i\le r$. From the residue parametrization theorem (p.[2]) we obtain

Corollary. *Let h be a polynomial with the degree* $\deg h\ge\deg f_1+\ldots+\deg f_n-n$ *and let* $s=\deg h-(\deg f_1+\ldots+\deg f_n)+n+1$
Then

$$\operatorname{Res}_c h/F=\operatorname{Res}_{\tilde{c}}\tilde{h}\big/\left(\tilde{f}_1,\ldots,\tilde{f}_{n-1},X_1^s\tilde{f}_n\right)=$$

$$\sum\nolimits_{1\le i\le r}\operatorname{res}_0\frac{\tilde{h}(\Phi_i(t))}{\left(\operatorname{Jac}\left(\tilde{f}_1,\ldots,\tilde{f}_{n-1},X_1^s\tilde{f}_n\right)\right)(\Phi_i(t))}\left(\frac{\left(\tilde{f}_n(\Phi_i(t))\right)'}{\tilde{f}_n(\Phi_i(t))}-\frac{s\mu_i}{t}\right)$$

References

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