# ON A MACROSCOPIC MODELLING OF NONSTATIONARY HEAT CONDUCTION PROBLEMS IN HEXAGONAL - TYPE RIGID CONDUCTORS

***Urszula Siedlecka, Ewaryst Wierzbicki***

*Institute of Mathematics and Computer Science, Czestochowa University of Technology*

**Abstract.** The aim of this contribution is to formulate a macroscopic model for the non-stationary heat transfer processes in an arbitrary hexagonal - type rigid conductor with isotropic local material properties. As a tool of modelling the tolerance averaging technique is taken into account [1]. It is shown that under certain conditions the obtained model equations are isotropic.

## 1. Introduction

Let us denote by $\Delta$ a certain regular hexagon in the $0x_1x_2$-plane. Let $c(\cdot,x_3)$ and $\mathbf{A}(\cdot,x_3)$ be $\Delta$-periodic fields interpreted as a specific heat and a heat conductivity tensor, respectively. In the framework of this note by a hexagonal-type rigid heat conductor we mean a conductor in which the distribution of material properties in the periodic cell $\Delta$ are invariant with respect to the rotations over angles $2\pi n/3$, $n = 0, \pm 1, \pm 2, \dots$ , in the $0x_1x_2$-plane. From the formal viewpoint, the set of all such rotations consists only of three elements, hence $n = 0,1,2$. Numbers 0,1,2 will be interpreted as the possible values $n$ in $2\pi n/3$-rotation and hence we deal with the well-known additive group $\Re_3 = \{0,1,2\}$ which in the subsequence considerations will be identified with the above rotations group. In the subsequent considerations without loss of generality it will be assumed that a specific heat and a heat conductivity tensor fields are constant in the $x_3$-direction and we will write $c(\mathbf{x}) = c(\mathbf{x}, x_3)$ and $\mathbf{A}(\mathbf{x}) = \mathbf{A}(\mathbf{x}, x_3)$ for every $(\mathbf{x},x_3) \in R^3$. Moreover, it is assumed that a heat conductivity tensor has a form of the following real matrix

$$\mathbf{A} = \begin{bmatrix} \overline{\mathbf{A}} & 0 \\ 0 & k_3 \end{bmatrix} \tag{1}$$

where $\alpha$, $\beta$ run over 1,2. The equation

$$\overline{\nabla} \cdot \left(\overline{\mathbf{A}} \cdot \overline{\nabla}\theta\right) + \partial\left(k_3 \partial\theta\right) - c\dot{\theta} + f = 0\,, \tag{2}$$

where $\theta$ and $f$ denote a temperature field and heat sources, respectively, describes a heat flow in a biperiodic composite under consideration (cf. [1] p. 102) and is a starting point to subsequent considerations. Coefficients of this equation are assumed to be $\Delta$-periodic piecewise constant functions of $\mathbf{x} = (x_1, x_2) \in R^2$. This is a motivation for formulation of approximated models of such composites in particular models with constant coefficients. Subsequently, we shall consider the tolerance averaged variant of equation (2), cf. [1]. The tolerance averaged models of composites, in contrast to homogenized models [2], are able to describe a dispersion phenomena and equations of such models have simple mathematical form. That is why, in the modeling of processes depending on time, tolerance averaged models seem to be more useful than the well-known asymptotic models. In tolerance averaged modelling of rigid conductors, the temperature field $\theta$ is a sum of averaged temperature $\theta^o$ and fluctuation part $\vartheta$ of temperature field, $\theta = \theta^o + \vartheta$. The investigation of an interrelation between the above fields is usually the crucial problem in every averaged approach to the modeling of a $\Delta$-periodic rigid conductor. In the tolerance averaged approach it is assumed that (summation convention over $A = 1,..., N$ holds)

$$\vartheta \cong g^A W^A \tag{3}$$

where $g^A$ are postulated *a priori* in every special problem $\Delta$-periodic shape functions, the relation $\cong$ describes a certain tolerance approximation, cf. [1], and $W^A$ are certain new fields which, together with the averaged temperature field $\theta^o$, are basic unknowns of the tolerance averaged model of the heat conductor under consideration. This model is governed by the following averaged equations (summation convention over $A,B = 1,..., N$ holds):

$$\begin{gathered}
\overline{\nabla} \cdot (\langle \overline{\mathbf{A}} \rangle \cdot \overline{\nabla} \theta^0 + \langle \overline{\mathbf{A}} \cdot \overline{\nabla} g^A \rangle W^A) - \langle c \rangle \dot{\theta} - \langle c g^A \rangle \dot{W}^A + \\
+ \partial(\langle k_3 \rangle \partial \dot{\theta}^0 - \langle k_3 g^A \rangle \partial W^A) = \langle f \rangle \\
\langle c g^A g^B \rangle \dot{W}^B + \langle \overline{\nabla} g^A \cdot \overline{\mathbf{A}} \cdot \overline{\nabla} g^B \rangle W^B + \langle \overline{\nabla} g^A \cdot \overline{\mathbf{A}} \rangle \cdot \overline{\nabla} \theta^0 + \\
+ \langle \langle c g^A \rangle \dot{\theta}^0 - \partial(\langle k_3 g^A g^B \rangle \partial W^B + \langle g^A k_3 \rangle \cdot \nabla \theta^0 = -\langle f g^A \rangle
\end{gathered} \tag{4}$$

where $g^A$, $A = 1,..., N$, are known and $\langle \cdot \rangle$ stands for the averaging operator [1].

The aim of this note is to investigate sufficient conditions, imposed on the specific heat, conductivity tensor fields and the set $G$ of shape functions, describing internal fluctuations of hexagonal periodicity cell, which imply the isotropic form of eqs(4).

## 2. Preliminaries

In the tolerance averaged model (4) of the hexagonal-type composite the shape functions depend on the thermal as well as material structure of the periodicity cell. Hence, we shall assume that the set of all shape functions taken into account in this model is invariant under the rotations group $\mathfrak{R}_3$. It can be proved that the heat conductivity tensor $\overline{\mathbf{A}}$ is isotropic, i.e. $\overline{\mathbf{A}} = k\mathbf{1}$. We are going to prove the following basic result of this note:

> ***Proposition***. *Under the above assumptions the tolerance averaged model* (4) *of the hexagonal-type rigid conductor is isotropic.*

Using terminology from the mathematical group theory, assumption mentioned above means that the group $\mathfrak{R}_3$ acts in the set of shape functions of the model (4) and divide this set into the disjoint sum $G_1 \cup G_2 \cup ... \cup G_n$ of orbits $G_k$, $k = 1,\dots,n$. A number of elements of every orbit is the same and is equal to the range of the group $\mathfrak{R}_3$. Hence, we can enumerate the shape functions, denoting by $g_r^k$ the $r$-th element of an orbit $G_k$ generated by an arbitrary distinguish shape function $g^k \in G_k$ and relation $g_r^k = g^k \circ r$. Now we shall introduce a concept of the first approximation model of the hexagonal-type rigid conductor. It will be a special kind of such model in which the group $\mathfrak{R}_3$ determines in the set of all shape functions only one $\mathfrak{R}_3$-orbit. Hence the set of shape functions is equal to this orbit, consists of three shape functions $g^k \in G$, $k = 0,1,2$, and will be denoted by $G$. From the physical viewpoint these shape functions are interpreted as the simplest functions describing the basis temperature fluctuations in the periodic cell. In the case of the first approximation the governing equations (4) take the form

$$\begin{gathered}
\overline{\nabla} \cdot (\langle \overline{\mathbf{A}} \rangle \cdot \overline{\nabla} \theta^0 + \langle \overline{\mathbf{A}} \cdot \overline{\nabla} g_r \rangle W^r) - \langle c \rangle \dot{\theta} - \langle c g_r \rangle \dot{W}^r + \\
+ \partial (\langle k_3 \rangle \partial \dot{\theta}^0 - \langle k_3 g_r \rangle \partial W^r) = \langle f \rangle \\
\langle c g_r g_s \rangle \dot{W}^s + \langle \overline{\nabla} g_r \cdot \overline{\mathbf{A}} \cdot \overline{\nabla} g_s \rangle W^s + \langle \overline{\nabla} g_r \cdot \overline{\mathbf{A}} \rangle \cdot \overline{\nabla} \theta^0 + \\
+ \langle \langle c g_r \rangle \dot{\theta}^0 - \partial (\langle k_3 g_r g_s \rangle \partial W^s + \langle g_r k_3 \rangle \cdot \nabla \theta^0 = -\langle f g_r \rangle
\end{gathered} \tag{5}$$

Similarly as in every problem of the tolerance modelling these three functions will be postulated *a priori* on the basis of the physical information about the modeled heat conductor.

For the sake of simplicity, in the subsequent consideration it will be assumed the first approximation of the model.

## 3. Results

Let us introduce vectors $\mathbf{t}^0 = (1,0)$, $\mathbf{t}^1 = (-1/2, \sqrt{3}/2,)$, $\mathbf{t}^2 = (-1/2, -\sqrt{3}/2,)$, $\tilde{\mathbf{t}}^0 = (0,1)$, $\tilde{\mathbf{t}}^1 = (-\sqrt{3}/2, -1/2)$, $\tilde{\mathbf{t}}^2 = (\sqrt{3}/2, -1/2)$ and denote $T^0 = (\mathbf{t}^0, \tilde{\mathbf{t}}^0)$,

$T^1 = (\mathbf{t}^1, \widetilde{\mathbf{t}}^1)$, $T^2 = (\mathbf{t}^2, \widetilde{\mathbf{t}}^2)$. The geometry of the hexagonal cell is illustrated in Figure 1. Since the set $G = \{g_r: r \in \Re_3\}$ as well as a set $T^1 = (T^0, T^1, T^2)$ of introduced above orthonormal basis are invariant over $\Re_3$-rotations the scalar products $h = \nabla g_r \cdot \mathbf{t}^r$ and $\widetilde{h} = \nabla g_r \cdot \widetilde{\mathbf{t}}^r$ are also invariant over rotations $r \in \Re_3$. Hence, we can introduce the decomposition

$$\nabla g_r = h\mathbf{t}^r + \widetilde{h}\widetilde{\mathbf{t}}^r \tag{6}$$

which leads to (here and in the subsequent considerations summation convention over $r,s = 0,1,2$ holds):

$$\begin{gathered}\langle \nabla g_r \cdot \mathbf{A} \cdot \nabla g_s \rangle = \langle \mathbf{A}hh \rangle : \mathbf{t}^r \otimes \mathbf{t}^s + \langle \mathbf{A}h\widetilde{h} \rangle : (\mathbf{t}^r \otimes \widetilde{\mathbf{t}}^s + \widetilde{\mathbf{t}}^r \otimes \mathbf{t}^s) + \langle \mathbf{A}\widetilde{h}\widetilde{h} \rangle : \widetilde{\mathbf{t}}^r \otimes \widetilde{\mathbf{t}}^s \\ \langle \mathbf{A} \cdot \nabla g_s \rangle = \langle \mathbf{A}h \rangle \cdot \mathbf{t}^s + \langle \mathbf{A}\widetilde{h} \rangle \cdot \widetilde{\mathbf{t}}^s \\ \langle \nabla g_r \cdot \mathbf{A} \rangle = \mathbf{t}^r \cdot \langle \mathbf{A}h \rangle + \widetilde{\mathbf{t}}^r \cdot \langle \mathbf{A}h\widetilde{h} \rangle\end{gathered} \tag{7}$$

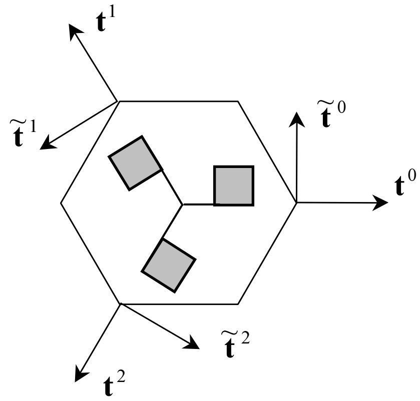


Fig. 1. The geometry of the hexagonal cell

Now we can start to describe the influence of invariance properties of the governing equations (5) under $\Re_3$-rotations. Namely, in contrast to (7), the coefficients $\langle \overline{\mathbf{A}} g_r \rangle$ for every $r \in \Re_3$ are invariant under $\Re_3$-rotations, coefficients $\langle \overline{\mathbf{A}} g_r g_s \rangle$ are invariant over $\Re_3$-rotations provided that $r \neq s$, $r,s \in \Re_3$, and coefficients $\langle \overline{\mathbf{A}} g_r g_r \rangle$ (no summation over $r$) are invariant over $\Re_3$-rotations for every $r \in \Re_3$. Hence, the formulas:

$$\begin{gathered}\langle \overline{\mathbf{A}} g_r g_{r+s} \rangle = \langle \overline{\mathbf{A}} g g_s \rangle \\ \langle \overline{\mathbf{A}} g_r \rangle = \langle \overline{\mathbf{A}} g \rangle\end{gathered} \tag{8}$$

hold for an arbitrary shape function $g \in G$ and $r,s \in \Re_3$. Let us introduce new unknowns

$$U = \sum_s W^s, \quad \mathbf{V} = \sum_s \mathbf{t}^s W^s, \quad \widetilde{\mathbf{V}} = \sum_s \widetilde{\mathbf{t}}^s W^s \tag{9}$$

It must be emphasized that the unknowns $\mathbf{V}$ and $\widetilde{\mathbf{V}}$ are interrelated by

$$\mathbf{V} = \in \cdot \widetilde{\mathbf{V}} \tag{10}$$

and that the formulas $(9)_{1,2}$ can be treated as a one to one interrelation between the internal fields $W^0$, $W^1$, $W^2$ and a new variables $U$ and $\mathbf{V} = (V_1, V_2)$. The inverse of this interrelation is given by

$$W^1 = \frac{1}{3}U + \frac{2}{3}V_1, \quad W^2 = \frac{1}{3}U - \frac{1}{3}V_1 + \frac{\sqrt{3}}{3}V_2, \quad W^3 = \frac{1}{3}U - \frac{1}{3}V_1 - \frac{\sqrt{3}}{3}V_2 \tag{11}$$

Applying the decomposition (6) into the first from eqs(5) we obtain

$$\begin{gathered}\overline{\nabla} \cdot \langle \overline{\mathbf{A}} \rangle \cdot \overline{\nabla} \theta^0 + \langle k_3 \rangle \partial^2 \theta + \langle \overline{\mathbf{A}} h \rangle : \nabla \mathbf{V} + \langle \overline{\mathbf{A}} \widetilde{h} \rangle : \nabla \widetilde{\mathbf{V}} + \langle k_3 g \rangle \partial^2 U - \langle c \rangle \dot{\theta}^0 - \\ - \langle cg \rangle \dot{U} + \langle f \rangle = 0\end{gathered} \tag{12}$$

Now we shall to apply the decomposition (6) into the second from eqs(5). Taking into account the invariant conditions explained above, equations (5) yield

$$\begin{gathered}\langle cg^r g^s \rangle \dot{W}^s - \langle k_3 g^r g^s \rangle \partial^2 W^s + \\ (\mathbf{t}^r \cdot \langle \overline{\mathbf{A}} hh \rangle + \widetilde{\mathbf{t}}^r \cdot \langle \overline{\mathbf{A}} h\widetilde{h} \rangle) \cdot \mathbf{V} + (\mathbf{t}^r \cdot \langle \overline{\mathbf{A}} h\widetilde{h} \rangle + \widetilde{\mathbf{t}}^r \cdot \langle \overline{\mathbf{A}} \widetilde{h}\widetilde{h} \rangle) \cdot \widetilde{\mathbf{V}} + \\ + (\mathbf{t}^r \cdot \langle \overline{\mathbf{A}} h \rangle + \widetilde{\mathbf{t}}^r \cdot \langle \overline{\mathbf{A}} h\widetilde{h} \rangle) \cdot \nabla \theta^0 - \langle k_3 g^r \rangle \partial^2 \theta^0 + \langle cg^r \rangle \dot{\theta}^0 = \langle fg^r \rangle\end{gathered} \tag{13}$$

Summing up over $r$ the above equations we conclude that

$$\dot{U} \sum_s \langle cgg^s \rangle - \partial^2 U \sum_s \langle k_3 gg^s \rangle - \partial^2 \theta^0 \sum_s \langle k_3 g^s \rangle + \dot{\theta}^0 \sum_s \langle cg^s \rangle = \langle fg^r \rangle \tag{14}$$

Moreover, multiplying Eq.(12) by $\mathbf{t}^r$ and then summing it up over $r$ we obtain

$$\begin{gathered}\dot{\mathbf{V}}(\langle cgg \rangle - \langle cgg^1 \rangle) - \dot{\widetilde{\mathbf{V}}} \langle cgg^1 \rangle - \partial^2 \mathbf{V}(\langle k_3 gg \rangle - \langle k_3 gg^1 \rangle) + \partial^2 \widetilde{\mathbf{V}} \langle cgg^1 \rangle + \\ + 1.5(\langle \overline{\mathbf{A}} hh \rangle + \in \cdot \langle \overline{\mathbf{A}} h\widetilde{h} \rangle) \cdot \mathbf{V} + 1.5(\langle \overline{\mathbf{A}} h\widetilde{h} \rangle + \in \cdot \langle \overline{\mathbf{A}} \widetilde{h}\widetilde{h} \rangle) \cdot \widetilde{\mathbf{V}} - \\ + 1.5(\langle \overline{\mathbf{A}} h \rangle + \in \cdot \langle \overline{\mathbf{A}} \widetilde{h} \rangle) \cdot \nabla \theta^0 = \sum_r \mathbf{t}^r \langle fg^r \rangle\end{gathered} \tag{15}$$

Similarly, multiplying Eq.(13) by $\widetilde{\mathbf{t}}^r$ and then summing it up over $r$ we obtain

$$\begin{aligned}&\dot{\widetilde{\mathbf{V}}}(\langle cgg\rangle - \langle cgg^1\rangle) - \dot{\mathbf{V}}\langle cgg^1\rangle - \partial^2\widetilde{\mathbf{V}}(\langle k_3 gg\rangle - \langle k_3 gg^1\rangle) + \partial^2\mathbf{V}\langle cgg^1\rangle + \\ &+1.5(-\in\cdot\langle\overline{\mathbf{A}}hh\rangle + \langle\overline{\mathbf{A}}h\widetilde{h}\rangle)\cdot\mathbf{V} + 1.5(-\in\cdot\langle\overline{\mathbf{A}}h\widetilde{h}\rangle + \langle\overline{\mathbf{A}}\widetilde{h}\widetilde{h}\rangle)\cdot\widetilde{\mathbf{V}} - \\ &+1.5(\langle -\in\cdot\overline{\mathbf{A}}h\rangle + \langle\overline{\mathbf{A}}\widetilde{h}\rangle)\cdot\nabla\theta^0 = \sum_r \widetilde{\mathbf{t}}^r\langle fg^r\rangle\end{aligned} \tag{16}$$

Summarizing the above results we conclude that equations (12), (14), (15), (16) can be treated as the system of independent equations for the averaged temperature field $\theta^0$ and the three new unknown fields - one scalar field $U$ and two vector fields $\mathbf{V}$ and $\widetilde{\mathbf{V}}$. Now we shall jump to the final conclusion that if the conductivity tensor $\overline{\mathbf{A}}$ is isotropic, i.e. $\overline{\mathbf{A}} = k\mathbf{1}$, then equations (12), (14), (15), (16) have constant coefficients and hence they are isotropic. This ends the proof of the basic result of the note.

## References


[1] Woźniak C., Wierzbicki E., Averaging techniques in thermomechanics of composite solids, Wydawnictwo Politechniki Częstochowskiej, Częstochowa 2000.

[2] Jikov V.V., Kozlov S.M., Oleinik O.A., Homogenization of differential operators and integral functionals, Springer Verlag, Berlin 1994.