FREE VIBRATIONS OF NON-UNIFORM RODS WITH ADDITIONALLY ATTACHED SPRING-MASS DISCRETE ELEMENTS

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Abstract. The paper deals with free vibration of a rod with a polynomial varying cross-section. The formulation and solution of the problem take into account an arbitrary number of spring-mass discrete elements additionally attached to the rod. The Green's function method was applied while solving the free vibration problem. The Green's functions corresponding to various cases of variable cross-sections of the rods were determined. The obtained exact solution was used in a numerical analysis of an influence of the parameter characterizing the system on its free vibration frequencies.

1. Introduction

Exact solutions for free longitudinal vibrations of uniform rods with classical boundary conditions are well known. The study results of vibration of non-uniform rods are presented in references [1-5]. Abrate in the paper [1] shows that for a class of non-uniform rods the equation of motion can be transformed into the equation of motion for a uniform rod. Kumar and Sujith [2] have presented a solution for a rod with a polynomial area variation and for a sinusoidal rod. In [3] Li presents research results in which for several functional relations between stiffness and mass distributions the governing differential equation for free vibration of the rod to the Bessel equation has been reduced. The solution approach was then used for determining the natural frequencies and mode shapes of multi-step non-uniform rods. In [4] Li *et all* present an approach which combines the recurrence formula with closed form solutions of one step rods in order to obtain the solution of the problem of a multi-step rod. Li [5] derived the solution for the free vibration of multi-step rods by using the transfer matrix method and solutions of one step rods. The authors of the discussed papers [1-5] obtained the exact solutions of the considered vibration problems.

In this paper the Green's function method is used for solving the free vibration problems of non-uniform rods with discrete spring-mass elements. The case of non-uniformity caused by a polynomial variation of the cross-section of the rod is considered. The necessary Green's function of the differential problem is determined by introducing new variables leading to the problem of searching a solution of a Bessel equation. The presented exact solution for one step non-uniform rods may be used to derive the solution for free longitudinal vibration of a multi-step rod.

2. Formulation and solution of the problem

Consider a non-uniform rod with discrete spring-mass elements r attached to it at points $\bar{x}_j$ (j = 1, 2,…, r) as shown in Figure 1. Free vibration of the rod is governed by the differential equation

$$\frac{\partial}{\partial x}\left[E\bar{A}(x)\frac{\partial v(x,t)}{\partial x}\right]-\rho\bar{A}(x)\frac{\partial^2 v(x,t)}{\partial t^2}=\sum_{j=1}^{r}\left[m_j\frac{\partial^2 v(x,t)}{\partial t^2}+k_j v(x,t)\right]\delta(x-\bar{x}_j) \quad (1)$$

where $\bar{A}(x)$ is the area of the cross-section at point x of the rod, E is the modulus of elasticity, ρ is the mass density of the rod material, $\delta(\ )$ is the Dirac delta function, m_j and k_j are the discrete masses and the stiffness coefficients of the springs, respectively. The function v satisfied homogeneous boundary conditions, which may be symbolically written in the following form:

$$\bar{\mathbf{B}}_0[v]\Big|_{x=0}=0,\quad \bar{\mathbf{B}}_1[v]\Big|_{x=L}=0 \quad (2)$$

where $\bar{\mathbf{B}}_0$ and $\bar{\mathbf{B}}_1$ are linear, spatial differential operators and L denotes the length of the rod.

In order to obtain the natural frequencies of the rod, ω, one assumes that

$$v(x,t)=u(x)e^{i\omega t} \quad (3)$$

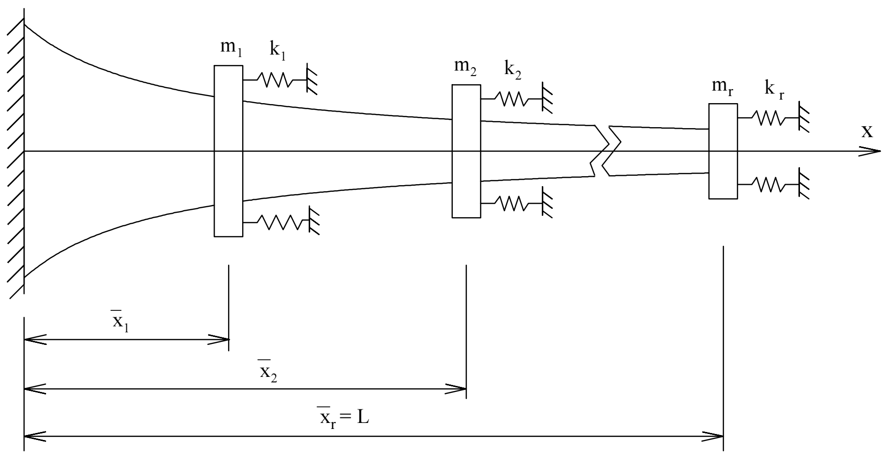

Fig. 1. A sketch of the non-uniform rod with the spring-mass discrete elements attached

By substituting equation (3) into equation (1) and introducing the non-dimensional co-ordinates and quantities: $\xi = \dfrac{x}{L}$, $\zeta_j = \dfrac{\overline{x}_j}{L}$, $U = \dfrac{u}{L}$, one obtains

$$\frac{d}{d\xi}\left[A(\xi)\frac{dU(\xi)}{d\xi}\right] + \Omega^2 A(\xi)U(\xi) = \sum_{j=1}^{r}\mu_j[\overline{\Omega}_j^2 - \Omega^2]U(\xi)\delta(\xi - \zeta_j) \tag{4}$$

where: $\Omega^2 = \dfrac{\rho L^2\omega^2}{E}$, $\overline{\Omega}_j^2 = \dfrac{\rho L^2\overline{\omega}_j^2}{E}$, $\overline{\omega}_j = \sqrt{\dfrac{k_j}{m_j}}$, $\mu_j = \dfrac{m_j}{\rho L A_0}$ for $j = 1,\dots,r$

$A(\xi) = \overline{A}(x)/A_0$ and $A_0 = \overline{A}(0)$. The boundary conditions (2) may be written in the following form

$$\mathbf{B}_0[U]\big|_{\xi=0} = 0, \quad \mathbf{B}_1[U]\big|_{\xi=1} = 0 \tag{5}$$

The solution of the problems (4)-(5) is obtained by using the properties of the Green's function G. The function G satisfied the boundary conditions (5) and the following differential equation

$$\mathbf{L}[G] = \delta(\xi - \zeta) \tag{6}$$

where

$$\mathbf{L} = \frac{d}{d\xi}\left[A(\xi)\frac{d}{d\xi}\right] + \Omega^2 A(\xi) \tag{7}$$

If the Green's function is known, then on the basis of equations (4)-(6) the following relationship is obtained

$$U(\xi) = \sum_{j=1}^{r}\mu_j[\overline{\Omega}_j^2 - \Omega^2]U(\zeta_j)G(\xi,\zeta_j) \tag{8}$$

By substituting $\xi = \zeta_i$, $i = 1,\dots,r$, successively into equation (8), one obtains a system of r homogeneous equations with unknowns $U(\zeta_j)$. For a non-trivial solution of the problem, the determinant of the coefficient matrix is set equal to zero, yielding the frequency equation

$$\left|a_{ij}(\Omega)\right| = 0 \tag{9}$$

where $a_{ij}(\Omega) = \mu_j\left(\Omega^2 - \overline{\Omega}_j^2\right)G(\zeta_i,\zeta_j) + \delta_{ij}$ and δ_{ij} is the Kronecker's delta. The equation (9), with the unknown Ω, is then solved numerically.

3. The Green's function for the vibration problem of a non-uniform rod

The Green's function $G(\xi,\zeta)$ of the differential problem considered satisfied the equation (6) and the boundary conditions (5). In order to determine it, one can observe first that the function may be written in the following form

$$G(\xi,\zeta) = G_0(\xi,\zeta) + G_1(\xi,\zeta)H(\xi-\zeta) \tag{10}$$

where H( ) is the Heaviside function, $G_0(\xi,\zeta)$ is the general solution and $G_1(\xi,\zeta)$ is a particular solution of the homogeneous equation

$$\mathbf{L}[U] = 0 \tag{11}$$

Moreover, the function $G_1(\xi,\zeta)$ satisfies the below conditions

$$G_1\big|_{\xi=\zeta} = 0, \quad \left.\frac{dG_1}{d\xi}\right|_{\xi=\zeta} = \frac{1}{A(\zeta)} \tag{12}$$

The general solution of the equation (11) may be determined by transforming the equation so that in the obtained equation the $A = A(\xi)$ is an independent variable. After suitable transformations the following equation is obtained

$$\left(\frac{dA}{d\xi}\right)^2 \frac{d^2U}{dA^2} + \frac{1}{A}\frac{d}{d\xi}\left[A\frac{dA}{d\xi}\right]\frac{dU}{dA} + \Omega^2 U = 0 \tag{13}$$

It is further assumed that $A(\xi) = (\alpha\xi + 1)^n$. In this case the equation (13) has the form

$$\frac{d^2U}{dA^2} + \left(2 - \frac{1}{n}\right)\frac{1}{A}\frac{dU}{dA} + \frac{1}{\alpha^2 n^2} A^{2\left(\frac{1}{n}-1\right)}\Omega^2 U = 0 \tag{14}$$

Next, new variables w, z [2] are introduced into the equation (14)

$$U = wA^{\gamma}, \quad z = \lambda A^{\sigma} \tag{15}$$

where: $\gamma = \dfrac{1-n}{2n}$, $\lambda = \dfrac{\Omega}{\alpha L}$, $\sigma = \dfrac{1}{n}$.

Then the Bessel's equation with $\upsilon = \dfrac{1-n}{2}$ is obtained

$$\frac{d^2w}{dz^2}+\frac{1}{z}\frac{dw}{dz}+\left(1-\frac{\upsilon^2}{z^2}\right)w=0 \tag{16}$$

The general solution of the equation (16) dependent on the value of the parameter υ, has the following form:

$$w=c_1J_\upsilon(z)+c_2Y_\upsilon(z) \quad \text{if } \upsilon \text{ is an integer} \tag{17}$$

$$w=c_1J_\upsilon(z)+c_2J_{-\upsilon}(z) \quad \text{if } \upsilon \text{ is not an integer} \tag{18}$$

Taking into account the equations (17)-(18) in equation (15) and (12) one obtains the Green's function:
– if υ is an integer:

$$\begin{gathered} G(\xi,\zeta)\ = A^\gamma(\xi)\,c_1(\zeta)\,J_\upsilon(z_\xi)+A^\gamma(\xi)\,c_2(\zeta)\,Y_\upsilon(z_\xi)+ \\ +M^{-1}[2J_\upsilon(z_{\xi-\zeta})Y_\upsilon(z_0)-A^{\sigma\upsilon}(\xi-\zeta)J_\upsilon(z_0)Y_\upsilon(z_{\xi-\zeta})]\,H(\xi-\zeta) \end{gathered} \tag{19}$$

– if υ is not an integer:

$$\begin{gathered} G(\xi,\zeta)\ = A^\gamma(\xi)\,c_1(\zeta)\,J_\upsilon(z_\xi)+A^\gamma(\xi)\,c_2(\zeta)J_{-\upsilon}(z_\xi)+ \\ +M^{-1}[2J_\upsilon(z_{\xi-\zeta})J_{-\upsilon}(z_0)-A^{\sigma\upsilon}(\xi-\zeta)J_\upsilon(z_0)J_{-\upsilon}(z_{\xi-\zeta})]\,H(\xi-\zeta) \end{gathered} \tag{20}$$

where: $z_\xi=z(\xi)$, $z_{\xi-\zeta}=z(\xi-\zeta)$, $z_0=z(0)$, $z_1=z(1)$ and

$$M=\alpha z_0[J_{-\upsilon}(z_0)(J_{\upsilon-1}(z_0)-J_{\upsilon+1}(z_0))-J_\upsilon(z_0)(J_{-\upsilon-1}(z_0)-J_{-\upsilon+1}(z_0))] \tag{21}$$

The constants c_1, c_2 occurring in equations (19), (20) are determined on the basis of boundary conditions (5). For the fixed-free rod ($G|_{\xi=0}=0,\ \left.\frac{dG}{d\xi}\right|_{\xi=1}=0$) the constants c_1, c_2 are the following:

$$c_1(\zeta)=-J_{-\upsilon}(z_0)\,N(\zeta)/D, \quad c_2(\zeta)=J_\upsilon(z_0)\,N(\zeta)/D \tag{22}$$

where

$$\begin{gathered} N(\zeta)=2A^{\sigma(1-\upsilon)}(1)\{2\nu[J_\upsilon(z_{1-\zeta})J_{-\upsilon}(z_0)-J_\upsilon(z_0)J_{-\upsilon}(z_{1-\zeta})]+ \\ +z_{1-\zeta}\,J_{-\upsilon}(z_0)[J_{\upsilon-1}(z_{1-\zeta})-J_{\upsilon+1}(z_{1-\zeta})]-z_{1-\zeta}\,J_\upsilon(z_0)[J_{-\upsilon-1}(z_{1-\zeta})-J_{-\upsilon+1}(z_{1-\zeta})]\} \end{gathered} \tag{23}$$

$$\begin{gathered} D=M\,A^{\sigma(1-\upsilon)}(1-\zeta)\{\ 2\nu[J_\upsilon(z_1)J_{-\upsilon}(z_0)-J_\upsilon(z_0)J_{-\upsilon}(z_1)]+ \\ +z_{1-\zeta}\,J_{-\upsilon}(z_0)[J_{\upsilon-1}(z_{1-\zeta})-J_{\upsilon+1}(z_{1-\zeta})]-z_{1-\zeta}\,J_\upsilon(z_0)[J_{-\upsilon-1}(z_{1-\zeta})-J_{-\upsilon+1}(z_{1-\zeta})]\} \end{gathered} \tag{24}$$

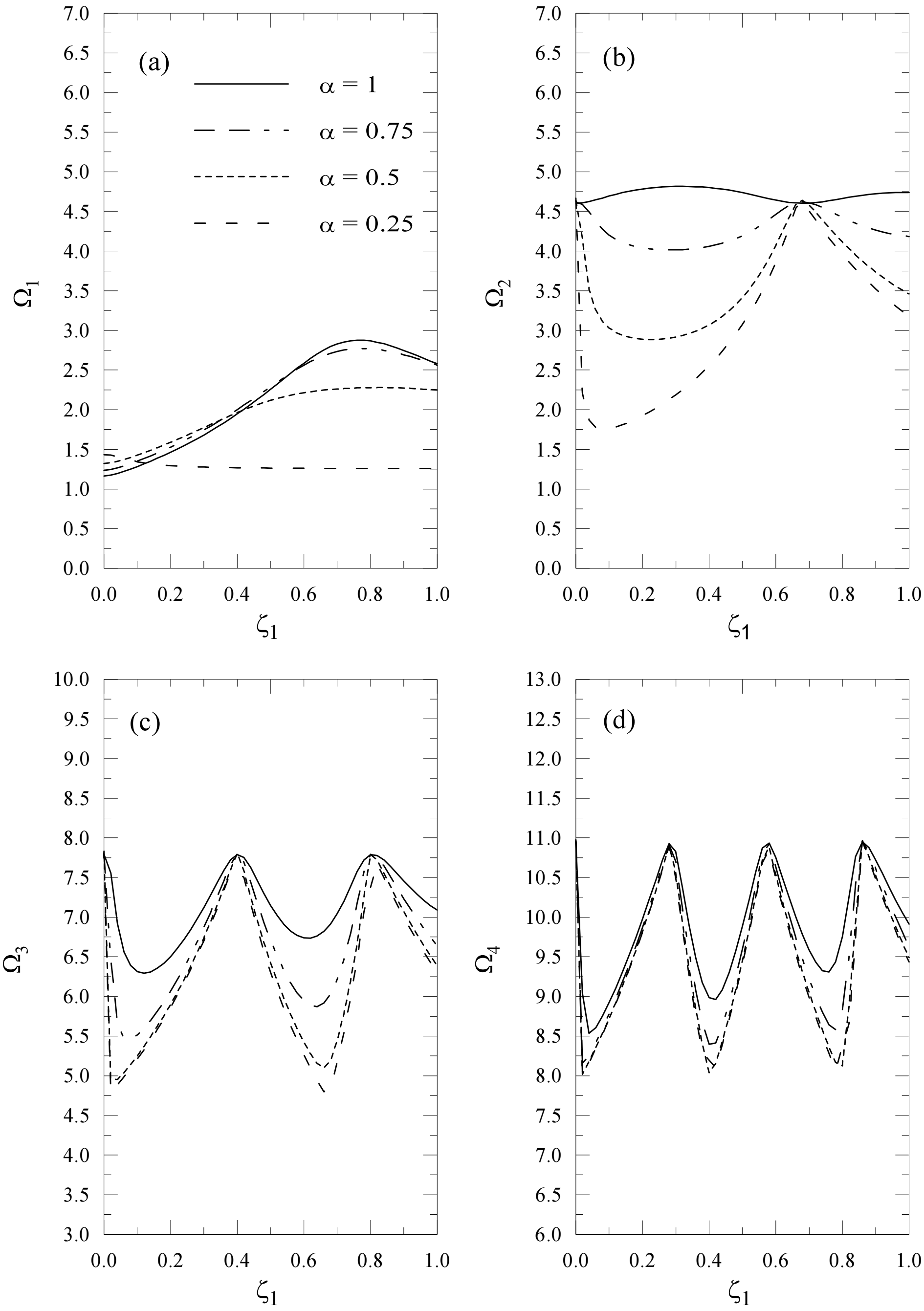

Fig. 2. Frequency parameter values Ω_i for the first four modes of vibration as a function of ζ_1 for the fixed-free rod with the cross-section $A(\xi) = (\alpha\,\xi + 1)^2$

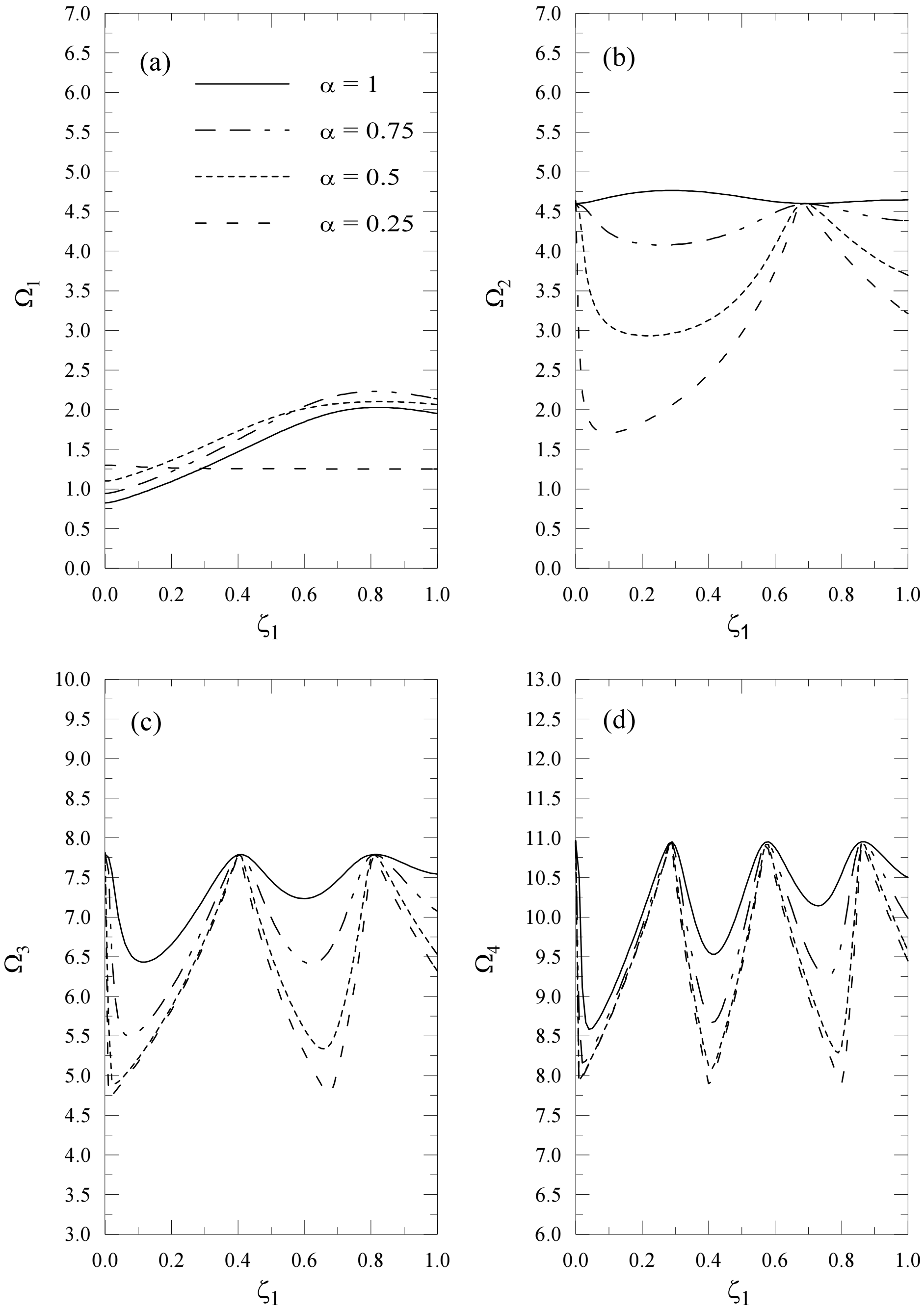

Fig. 3. Frequency parameter values Ω_i for the first four modes of vibration as a function of ζ_1 for the fixed-free rod with the cross-section $A(\xi) = (\alpha\,\xi + 1)^4$

4. Numerical example

Free vibration of a rod with a polynomial varying cross-section: $A(\xi) = (\alpha\xi + 1)^n$ for n = 2; 4, is considered. A discrete spring-mass element with the concentrated mass m and spring stiffness k is mounted at the point $\xi = \zeta_1$ of the rod. Basing on the equation (9) the frequency equation of the system has the following form

$$\mu_1\left(\Omega^2 - \overline{\Omega}_1^2\right) G(\zeta_1, \zeta_1; \Omega) + 1 = 0 \tag{25}$$

The non-dimensional free vibration frequencies Ω_i are numerically calculated as functions of the position ζ_1 of the discrete element. One assumes that: $\overline{\Omega}_1 = \sqrt{k_1 / m_1} = 5.0$. The curves of $\Omega_i = \Omega_i(\zeta_1)$, i = 1,…,4, are presented for various values of the parameter α, which characterise non-uniformity of the rod in Figures 2 and 3. The results of computations prove that non-uniformity of the rod can cause essential changes in the free vibration frequencies of the system.

Conclusions

The exact solution for the free vibration problem of a non-uniform rod with discrete elements has been obtained by the application of the Green's function method. One derives the Green's function corresponding to the differential problem considered. The numerical examples have shown the effect of selected parameters characterizing non-uniformity on the free vibration frequencies of the rod. Although a rod with one spring-mass element is considered as an example here, the solution can be used for any rod with an arbitrary number of discrete elements of this type.

References

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