ON SYMMETRIC POISSON STRUCTURE AND LIE BRACKET IN LINEAR ALGEBRES

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Abstract. In the paper the symmetric Poisson structure on linear has been applied. A connection of this structure with Lie bracket has been detined.

Let V be a linear algebra over R and let

$$A : V \times \mathrm{V} \to V$$

be a skew - symmetric 2-linear mapping satisfying the conditions

$$A(\alpha \cdot \beta, \gamma) = \alpha A(\beta, \gamma) + \beta A(\alpha, \beta) \tag{i}$$

$$A(A(\alpha, \beta), \gamma) + A(A(\gamma, \alpha), \beta) + A(A(\beta, \gamma), \alpha) = 0 \tag{ii}$$

for any $\alpha, \beta, \gamma \in V$.

The mapping A is said to be a Poisson structure on V and the pair (V, A) we will called a Poisson linear algebra.

From definition it follows that for any $\alpha \in V$ the mapping

$$D_\alpha := A(\cdot, \alpha) : V \to V$$

is a derivation of the algebra V.

It is easily to prove.

Proposition 1. The set $D(V)$ of all derivations D_α of V is a linear space over R. Moreover $D(V)$ is a Lie algebra with the Lie bracket given by

$$[D_\alpha, D_\beta] = D_\alpha \cdot D_\beta - D_\beta \cdot D_\alpha \tag{2}$$

for any $D_\alpha, D_\beta \in D(V)$.

Proposition 2. For any $\alpha, \beta \in V$

$$[D_\alpha, D_\beta] = D_{A(\beta,\alpha)} \tag{3}$$

An element $\alpha \in V$ is said to be a Casimir element of V with respect to A, if $A(\alpha, \beta) = 0$ for any $\beta \in V$. The set of all Casimir element of V with respect to A we denote by V_C^A. Evidently the pair (V, A) is a Lie algebra and V_C^A is its ideal.

Now let $T : V \to V$ be a mapping satisfying the condition

$$A(T(\alpha), \beta) = -A(\alpha, T(\beta)) \tag{4}$$

for any $\alpha, \beta \in V$.

Proposition 3. A mapping $T : V \to V$ satisfying the condition (4) has the following properties:

$$T(\alpha + \beta) = T(\alpha) + T(\beta) + \gamma \tag{i}$$

$$T(x \cdot \alpha) = xT(\alpha) + \delta \tag{ii}$$

for any $\alpha, \beta \in V$ and $x \in V_C^A$, where γ and δ are some elements of V_C^A.

Proof. For any $\alpha, \beta \in V$ by (4) we have.

$$\begin{aligned} A(T(\alpha + \beta), \gamma) &= -A(\alpha + \beta, T(\gamma)) = -A(\alpha, T(\gamma)) - A(\beta, T(\gamma)) = \\ &= A(T(\alpha), \gamma) + A(T(\beta), \gamma) \end{aligned}$$

Hence

$$A(T(\alpha + \beta) - T(\alpha) - T(\beta), \gamma) = 0$$

which gives

$$T(\alpha + \beta) = T(\alpha) + T(\beta) + \gamma$$

for some $\gamma \in V_C^A$.

Similarly we have

$$\begin{aligned} A(T(x\alpha), \beta) &= -A(x\alpha, T(\beta)) = -xA(\alpha, T(\beta)) = \\ &= xA(T(\alpha), \beta) = A(xT(\alpha), \beta) \end{aligned}$$

Hence

$$A(T(x, \alpha) - xT(\alpha), \beta) = 0$$

which gives $T(x\alpha) = xT(\alpha) + \delta$ for any $\alpha \in V$, $x \in V_C^A$ where δ is some element of V_C^A.

One can easily top prove

Proposition 4. A mapping $T:V\to V$ satisfying the condition (4) satisfies also the conditions.

$$A(T^n(\alpha),\beta)=(-1)^n A(\alpha,T^n(\beta)) \quad \text{(i)}$$

$$T^n(\alpha+\beta)=T^n(\alpha)+T^n(\beta)+\gamma \quad \text{(ii)}$$

$$T^n(x\alpha)=xT^n(\alpha)+\delta \quad \text{(iii)}$$

for any $\alpha,\beta\in V$, $x\in V_C^A$ and $n\in N$, where γ and δ are some elements of V_C^A.

Proposition 5. If $\alpha\in V_C^A$ then $T(\alpha)\in V_C^A$. In consequence V_C^A is a *T*-invariant linear subspace of the linear space *V*.

Proof. Let $\alpha\in V_C^A$, then for any $\beta\in V$ $A(\alpha,\beta)=0$, for any $\beta\in V$. Therefore $T(\alpha)\in V_C^A$.

Let us put

$$S(\alpha,\beta)=A(T(\alpha),\beta) \quad (5)$$

for any $\alpha,\beta\in V$.

Evidently the formula (5) defines a 2-linear mapping $S:V\times V\to V$.

Lemma 6. The mapping *S* defined by (5) is symmetric one.

Prof. From (4) and (5) it follows

$$S(\alpha,\beta)=A(T(\alpha),\beta)=-A(\alpha,T(\beta))=A(T(\beta),\alpha)=S(\beta,\alpha)$$

for any $\alpha,\beta\in V$.

Now we will prove

Proposition 7. The mapping *S* defined by (5) satisfies the identities

$$S(T(\alpha),\beta)=-s(\alpha,T(\beta)) \quad \text{(i)}$$

$$S(\alpha\cdot\beta,\gamma)=\alpha S(\beta,\gamma)+\beta S(\alpha,\gamma) \quad \text{(ii)}$$

$$S(S(T(\alpha),\beta),\gamma)+S(S(T(\gamma),\alpha),\beta)+S(S(T(\beta),\gamma),\alpha)=0 \quad \text{(iii)}$$

for any $\alpha,\beta,\gamma\in V$.

Proof. (i). Using (4) and (5) we get

$$S(\alpha, T(\beta)) = A(TT(\alpha), T(\beta)) = -A(T(\beta), T(\alpha)) = -S(T(\alpha), \beta)$$

for any $\alpha, \beta \in V$.

(ii) From (4) and (5) as well as from definition of *A* we get

$$\begin{aligned} S(\alpha \cdot \beta, \gamma) &= -A(\alpha \cdot \beta T(\gamma)) = \\ &= -\alpha A(\beta, T(\gamma)) - \beta A(\alpha, T(\gamma)) = \alpha S(\beta, \gamma) + \beta S(\alpha, \gamma) \end{aligned}$$

for any $\alpha, \beta, \gamma \in V$.

(iii) Analogically we get

$$\begin{aligned} &A(A(T(\alpha), T(\beta)), T(\gamma)) + A(A(T(\gamma), T(\alpha)), T(\beta)) + \\ &+ A(A(T(\beta), T(\gamma)), T(\alpha)) = -S(A(T(\alpha), T(\beta)), \gamma) + \\ &- S(A(T(\gamma), T(\alpha)), \beta) - S(A(T(\beta), T(\gamma)), \alpha) = S(S(T(\alpha), \beta), \gamma) + \\ &+ S(S(T(\gamma), \alpha), \beta) + S(S(T(\beta), \gamma), \alpha) = 0 \end{aligned}$$

for any $\alpha, \beta, \gamma \in V$.

So, we may accept

Def. 1. A mapping *S*, defined by (5) is said to be a symmetric Poisson structure on a linear algebra *V* over *R*.

From proposition 5 (ii) it follows that for any $\alpha \in V$ the mapping

$$\delta_\alpha = S(\cdot, \alpha) : V \to V \tag{6}$$

is a derivation of the algebra *V*.

Proposition 8. The set $\Delta(V)$ of all derivations δ_α of $\alpha \in V$, is a linear space over *R*. Moreover $\Delta(V)$ is a Lie algebra with a Lie bracket given by

$$\lfloor \delta_\alpha, \delta_\beta \rfloor = \delta_\alpha \cdot \delta_\beta - \delta_\beta \cdot \delta_\alpha$$

for any $\delta_\alpha, \delta_\beta \in \Delta(V)$.

From (1), (5) and (6) it follows the relation

$$\delta_\alpha = -D$$

for any $\alpha \in V$ and consequently $\lfloor \delta_\alpha, \delta_\beta \rfloor \cdot T = \delta_{S(T(\alpha), \beta)}$ for any $\alpha, \beta \in V$.

Def. 2. An element $\alpha \in V$ is said to be a Casimir element of *V* with respect to *S*, if $S(\alpha, \beta) = 0$ for any $\beta \in V$.

The set of all Casimir elements of V with respect to S we denote by V_C^S.
We shall prove.

Lemma 9. If $\alpha \in V_C^A$ then $T(\alpha) \in V_C^S$.

Proof. Let $\alpha \in V_C^A$. By Proposition 5 $T(\alpha) \in V_C^S$. Hence by (5)

$$S(\alpha, \beta) = A(T(\alpha), \beta) = 0$$

for any $\beta \in V$. Therefore $\alpha \in V_C^S$.

Lemma 10. $\alpha \in V_C^S$ in and only if $T(\alpha) \in V_C^A$.

Proof. It follows from $S(\alpha, \beta) = A(T(\alpha), \beta)$ for $\beta \in V$.

Lemma 11. If $\alpha \in V_C^S$ then $T(\alpha) \in V_C^S$.

Proof. Let $\alpha \in V_C^S$ then $S(\alpha, \beta) = 0$ for any $\beta \in V$. Hence $S(\alpha, TT(\beta)) = = -S(T(\alpha), \beta) = 0$ for any $\beta \in V$. Therefore $T(\alpha) \in V_C^S$.

Corollary 12. V_C^S is T-invariant subspace of the linear space V.

Evidently, if $T : V \to V$ is onto then $V_C^S = V_C^A$. In general case there is the inclusion $V_C^S \supset V_C^A$.

Let us observe also that (V, S) is an algebra, which we shall call a symmetric Lie algebra. Of course V_C^S is an ideal of this algebra.

Let $T : V \to V$ be a mapping satisfying the condition

$$A(\alpha, T(\beta)) = -A((\alpha), \beta)$$

for any $\alpha, \beta \in V$. This mapping induces the mapping

$$T_* : D(V) \to D(V) \tag{7}$$

given by

$$T_*(D_\alpha) = D_{T(\alpha)} \tag{8}$$

for any $D_\alpha \in D(V)$.

Lemma 13. The mapping T_* Defined by (8) satisfies the condition

$$\lfloor T_* D_\alpha, D_\beta \rfloor = -\lfloor D_\alpha, T_* D_\beta \rfloor \tag{9}$$

for any $D_\alpha, D_\beta \in D(V)$.

Proof. Using from (5) we get for any $D_\alpha, D_\beta \in D(V)$.

$$\lfloor T_* D_\alpha, D_\beta \rfloor = \lfloor D_{T(\alpha)}, D_\beta \rfloor = D_{A(\beta, T(\alpha))} = -D_{A(T(\beta),\alpha)} = \\ = -\lfloor D_\alpha, D_{T(\beta)} \rfloor = -\lfloor D_\alpha, T_* D_\beta \rfloor$$

Now let us put

$$\lfloor (D_{\alpha 0}, D_\beta) \rfloor = \lfloor T_* D_\alpha, D_\beta \rfloor \tag{10}$$

for any $D_\alpha, D_\beta \in D(V)$.

It is easily to observe that the formula (10) defines a 2-linear mapping.

$$[(\cdot,\cdot)] : D(V) \times D(V) \to D(V)$$

Lemma 14. The mapping $[(\cdot,\cdot)]$ defined by (10) is a symmetric one.

Proof. By (9) and (10) we have

$$\lfloor (D_\alpha, B_\beta) \rfloor = \lfloor T_* D_\alpha, D_\beta \rfloor = -\lfloor D_\alpha, T_* D_\beta \rfloor = \lfloor T_* D_\beta, D_\alpha \rfloor = \lfloor (D_\beta, D_\alpha) \rfloor$$

for any $D_\alpha, D_\beta \in D(V)$.

Proposition 15. The mapping $[(\cdot,\cdot)]$ defined by (10) the following properties

$$\lfloor (T_* D_\alpha, D_\beta) \rfloor = -\lfloor (D_\alpha, T_* D_\beta) \rfloor \tag{i}$$

$$\lfloor (\lfloor (T_* D_\alpha, D_\beta) \rfloor, D_\gamma) \rfloor + \lfloor (\lfloor (T_* D_\gamma, D_\alpha) \rfloor, D_\beta) \rfloor + \lfloor (\lfloor (T_* D_\beta, D_\gamma) \rfloor, D_\alpha) \rfloor = 0 \tag{ii}$$

for any $D_\alpha, D_\beta \in D(V)$.

Proof. (i) From (9) and (10) we get for any $D_\alpha, D_\beta \in D(V)$

$$\lfloor (D_\alpha, T_* D_\beta) \rfloor = \lfloor T_* D_\alpha, T_* D_\beta \rfloor = -[T_* D\beta, T_* D_\alpha] = -\lfloor T_* D_\alpha, D_\beta \rfloor$$

(ii) Now for any $D_\alpha, D_\beta, D_\gamma \in D(V)$ we get

$$\lfloor \lfloor T_* D_\alpha, T_* B_\beta \rfloor, T_* D_\gamma \rfloor + \lfloor \lfloor T_* D_\gamma, T_* B_\alpha \rfloor, T_* D_\beta \rfloor + \lfloor \lfloor T_* D_\beta, T_* B_\gamma \rfloor, T_* D_\alpha \rfloor = \\ = -\lfloor (\lfloor T_* D_\alpha, T_* D_\beta \rfloor, D_\gamma) \rfloor - \lfloor (\lfloor T_* D_\gamma, T_* D_\alpha \rfloor, D_\beta) \rfloor - \lfloor (\lfloor T_* D_\beta, T_* D_\gamma \rfloor, D_\alpha) \rfloor = \\ = \lfloor (\lfloor (T_* D_\alpha, D_\beta) \rfloor, D_\gamma) \rfloor + \lfloor (\lfloor (T_* D_\gamma, D_\alpha) \rfloor, D_\beta) \rfloor + \lfloor (\lfloor (T_* D_\beta, D_\gamma) \rfloor, D_\alpha) \rfloor = 0$$

So, we shall accept

Def. 3. The mapping $[(\cdot,\cdot)]$ defined by (10) is said to be a symmetric Lie bracket.

It is easily to prove.

Proposition 16. The mapping $T_* : D(V) \times D(V) \to D(V)$ defined by (8) is a linear one over V_C^S.

Let (*V, A*) be a Poisson linear algebra and let *D*(*V*) denotes the Lie algebra of all derivations of *V* defined by (1). Now, let

$$\psi : D(V) \to D(V)$$

be a mapping satisfying the condition

$$\lfloor \psi(D_\alpha), D_\beta \rfloor = -\lfloor D_\alpha \psi(D_\beta) \rfloor \tag{11}$$

for any $D_\alpha, D_\beta \in D(V)$.

One can easily prove

Lemma 17. A mapping $\psi : D(V) \to D(V)$ satisfying the condition (11) is a linear one over *R*.

References

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