THE JACOBIANS OF LOWER DEGREES

Grzegorz Biernat

Institute of Mathematics and Computer Science, Czestochowa University of Technology

Abstract. In the present paper we give some relation of the number of zeros of a polynomial mapping in $\boldsymbol{C}^2$ with a jacobian of non-maximal degree and the number of branches at infinity of one coordinate of this mapping.

1. Auxiliary facts

Let $l_\infty = V(T_0)$ denote a line at infinity in the projective complex space $\boldsymbol{P}^2$ (with homogeneous coordinates $T_0 : T_1 : T_2$). Further it will be called infinity. If $a \in l_\infty$ then by $\widetilde{a} \in \boldsymbol{C}^2$ we denote the canonical image of the point a in affine part $\boldsymbol{P}^2 \setminus V(T_1) \cong \boldsymbol{C}^2$. For a polynomial h of two variables, $\widetilde{h}$ signifies a suitable dehomogenization of the homogenization of the polynomial h. So, we have $\widetilde{h}(X_1, X_2) = X_1^{\deg h} h(1/X_1, X_2/X_1)$.

Let f_1, f_2 and g be polynomials of two variables and let C_1, C_2 be the closures, respectively, of the curves $V(f_1), V(f_2)$ in the space $\boldsymbol{P}^2$. Assume further that polynomials f_1 and f_2 are different from constants and write $n_1 = \deg f_1$, $n_2 = \deg f_2$. We denote by $J_f = \mathrm{Jac}(f_1, f_2)$ (respectively, $J_{\widetilde{f}} = \mathrm{Jac}(\widetilde{f}_1, \widetilde{f}_2)$) the jacobian of the mapping $f = (f_1, f_2)$ (respectively, $\widetilde{f} = (\widetilde{f}_1, \widetilde{f}_2)$).

Fact 1. If $\deg J_f < \deg f_1 + \deg f_2 - 2$, then $(C_1 \cup C_2) \cap l_\infty = (C_1 \cap C_2) \cap l_\infty$.

Proof. Let f_1^+, f_2^+ be the leading forms of the polynomials f_1, f_2, respectively. Put $f^+ = (f_1^+, f_2^+)$. Since the degree of the jacobian J_f is not maximal, then the jacobian $J_{f^+} = 0$. It means that the homogeneous polynomials f_1^+, f_2^+ are algebraically independent. Thus, there is a polynomial h of two complex variables of positive degree without constant term such that $h \circ f^+ = 0$. Let

$$h(Y_1, Y_2) = \sum\nolimits_{i,j} c_{ij} Y_1^{\alpha_i} Y_2^{\beta_j}, \text{ where } c_{ij} \neq 0, \ \alpha_i + \beta_j \geq 1$$

For an arbitrary point $(a,b) \in \boldsymbol{C}^2 \setminus \{(0,0)\}$ we have

$$h\left(t^{n_1} f_1^+(a,b), t^{n_2} f_2^+(a,b)\right) = 0 \text{ for } t \in \boldsymbol{C} \tag{1}$$

If the point $(0:a:b) \in C_1 \cap l_\infty$, then $f_1^+(a,b) = 0$ and (1) reduces to identity

$$\sum_j d_j \left(f_2^+(a,b)\right)^{\beta_j} t^{\beta_j n_2} = 0, \text{ where } d_j \neq 0,\ \beta_j \geq 1,\ t \in \boldsymbol{C}$$

It means that $f_2^+(a,b) = 0$ and the point $(0:a:b) \in C_2 \cap l_\infty$. Analogously, if $(0:a:b) \in C_2 \cap l_\infty$, then $(0:a:b) \in C_1 \cap l_\infty$. This ends the proof.

Assume further that the polynomials f_1 and f_2 have not common factors of positive degrees and the polynomial f_1 has not irreducible multiples factors. Then the canonical image $\widetilde{a}$ of a point $a \in (C_1 \cap C_2) \cap l_\infty$ is an isolated zero of the mapping $\widetilde{f}$ and the germ $\left(\widetilde{f}_1\right)_{\widetilde{a}}$ of the function $\widetilde{f}_1$ in the point $\widetilde{a}$ has reduced decomposition [2]. Let

$$\left(\widetilde{f}_1\right)_{\widetilde{a}} = h_1 \ldots h_k \tag{2}$$

be suitable decomposition of the germ $\left(\widetilde{f}_1\right)_{\widetilde{a}}$ into irreducible single factors in the ring of the germs of holomorphic functions in the point $\widetilde{a}$. Write

$$\mu_i = \operatorname{ord}_{\widetilde{a}} h_i \text{ and } \kappa_i = \operatorname{mult}_{\widetilde{a}}(h_i, f_2) \text{ for } 1 \leq i \leq k$$

Fact 2. *If* $\kappa_i - n_2\mu_i \neq 0$ *for* $1 \leq i \leq k$, then the germ $\left(\widetilde{J}_f\right)_{\widetilde{a}}$ does not vanish identically on the set of zeros of all factors in the decomposition (2). In particular $J_f \neq 0$.

Proof. Assume contrary that for parametrization $\Phi_{i_0}(t) = \left(t^{\mu_{i_0}}, \varphi_{i_0}(t)\right)$ of zeros of the factor h_{i_0} in the decomposition (2) we have $\widetilde{J}_f\left(\Phi_{i_0}(t)\right) = 0$. Then according to the formula (*) in [1] we have

$$n_2 \widetilde{f}_2\left(\Phi_{i_0}(t)\right) \frac{\partial \widetilde{f}_1}{\partial X_2}\left(\Phi_{i_0}(t)\right) + t^{\mu_{i_0}} J_{\widetilde{f}}\left(\Phi_{i_0}(t)\right) = 0 \tag{3}$$

From another hand we have also

$$\left(\widetilde{f}_2\left(\Phi_{i_0}(t)\right)\right)' \frac{\partial \widetilde{f}_1}{\partial X_2}\left(\Phi_{i_0}(t)\right) + \left(t^{\mu_{i_0}}\right)' J_{\widetilde{f}}\left(\Phi_{i_0}(t)\right) = 0 \tag{4}$$

The equalities (3) and (4) have not zero solution, so

$$n_2\tilde{f}_2\left(\Phi_{i_0}(t)\right)\left(t^{\mu_{i_0}}\right)' - t^{\mu_{i_0}}\left(\tilde{f}_2\left(\Phi_{i_0}(t)\right)\right)' = 0$$

and

$$\frac{\left(\tilde{f}_2\left(\Phi_{i_0}(t)\right)\right)'}{\tilde{f}_2\left(\Phi_{i_0}(t)\right)} = \frac{n_2\left(t^{\mu_{i_0}}\right)'}{t^{\mu_{i_0}}}$$

Simple integration gives $\kappa_i = n_2\mu_i$, which contradicts assumption.

2. Basic fact

Assume that the polynomial f_1 is irreducible and $\deg J_f < \deg f_1 + \deg f_2 - 2$. Let g_1 denotes the genus of the curve C_1 and let $a_1, \ldots, a_s$ be the zeros at infinity of the mapping *f*. According to the Fact 1 we infer that these zeros are exactly the points at infinity of the curve C_1. In each point $\tilde{a}_k$ we have reduced decomposition

$$\left(\tilde{f}_1\right)_{\tilde{a}_k} = h_1^{(k)} \ldots h_{r_k}^{(k)} \quad \text{for} \quad 1 \le k \le s \tag{5}$$

where r_k denotes the number of branches of the curve C_1 in the point a_k at infinity. Write

$$\mu_j^{(k)} = \operatorname{ord}_{\tilde{a}_k} h_j^{(k)} \quad \text{and} \quad \kappa_j^{(k)} = \operatorname{mult}_{\tilde{a}_k}\left(h_j^{(k)}, \tilde{f}_2\right) \quad \text{for } 1 \le j \le r_k$$

Fact 3. *Let p be the number of zeros of the mapping f and q the number of zeros of the mapping (f_1, J_f) with respect of the multiplicity. If $\kappa_j^{(k)} - n_2\mu_j^{(k)} \ne 0$ for $1 \le k \le s$ and $1 \le j \le r_k$, then $p + \sum_{k=1}^{s} r_k \le q + 2(1 - g_1)$. Moreover the number $p + \sum_{k=1}^{s} r_k - q$ is even.*

Proof. For every point $\tilde{a}_k$ define non-negative integer

$$\delta_k = \frac{1}{2}\left(M_k + r_k - 1\right)$$

where M_k is the Milnor number of the curve $V(\tilde{f}_1)$ at the point $\tilde{a}_k$ [3]. Summing we have

$$\sum_{k=1}^{s}\delta_k = \frac{1}{2}\sum_{k=1}^{s} M_k + \frac{1}{2}\left(\sum_{k=1}^{s} r_k - s\right) \tag{6}$$

For every function $h_j^{(k)}$ from the decomposition (5) denote by $\Phi_j^{(k)}(t)=\left(t^{\mu_j^{(k)}},\varphi_j^{(k)}(t)\right)$ the parametrization of its zeros. From the formula (*) in [1] it follows that

$$\mu_j^{(k)}t^{\mu_j^{(k)}\sigma}\tilde{J}_f\left(\Phi_j^{(k)}(t)\right)=-n_2\mu_j^{(k)}\tilde{f}_2\left(\Phi_j^{(k)}(t)\right)\frac{\partial\tilde{f}_1}{\partial X_2}\left(\Phi_j^{(k)}(t)\right)-\mu_j^{(k)}t^{\mu_j^{(k)}}J_{\tilde{f}}\left(\Phi_j^{(k)}(t)\right) \quad (7)$$

where $\sigma=n_1+n_2-2-\deg J_f\geq 1$. From another hand we have

$$\mu_j^{(k)}t^{\mu_j^{(k)}-1}J_{\tilde{f}}\left(\Phi_j^{(k)}(t)\right)=-\left(\tilde{f}_2\left(\Phi_j^{(k)}(t)\right)\right)'\frac{\partial\tilde{f}_1}{\partial X_2}\left(\Phi_j^{(k)}(t)\right)$$

so

$$\mu_j^{(k)}t^{\mu_j^{(k)}}J_{\tilde{f}}\left(\Phi_j^{(k)}(t)\right)=-t\left(\tilde{f}_2\left(\Phi_j^{(k)}(t)\right)\right)'\frac{\partial\tilde{f}_1}{\partial X_2}\left(\Phi_j^{(k)}(t)\right) \quad (8)$$

From (7) and (8) we have

$$\mu_j^{(k)}t^{\mu_j^{(k)}\sigma}\tilde{J}_f\left(\Phi_j^{(k)}(t)\right)=\frac{\partial\tilde{f}_1}{\partial X_2}\left(\Phi_j^{(k)}(t)\right)\left(-n_2\mu_j^{(k)}\tilde{f}_2\left(\Phi_j^{(k)}(t)\right)+t\left(\tilde{f}_2\left(\Phi_j^{(k)}(t)\right)\right)'\right)$$

Since $\tilde{f}_2\left(\Phi_j^{(k)}(t)\right)=ct^{\kappa_j^{(k)}}+$ higher terms, where $c\neq 0$ and $\kappa_j^{(k)}-n_2\mu_j^{(k)}\neq 0$, the order of the second factor on the right side of the above equality is equal $\kappa_j^{(k)}$. Taking into account of both sides we have

$$\mu_j^{(k)}\sigma+\operatorname{ord}_0\tilde{J}_f\left(\Phi_j^{(k)}(t)\right)=\operatorname{ord}_0\frac{\partial\tilde{f}_1}{\partial X_2}\left(\Phi_j^{(k)}(t)\right)+\kappa_j^{(k)}$$

and summing

$$\sigma\sum_{j=1}^{r_k}\mu_j^{(k)}+\sum_{j=1}^{r_k}\operatorname{ord}_0\tilde{J}_f\left(\Phi_j^{(k)}(t)\right)=\sum_{j=1}^{r_k}\operatorname{ord}_0\frac{\partial\tilde{f}_1}{\partial X_2}\left(\Phi_j^{(k)}(t)\right)+\sum_{j=1}^{r_k}\kappa_j^{(k)}$$

so

$$\sigma\operatorname{ord}_{\tilde{a}_k}\tilde{f}_1+\operatorname{mult}_{\tilde{a}_k}\left(\tilde{f}_1,\tilde{J}_f\right)=\operatorname{mult}_{\tilde{a}_k}\left(\tilde{f}_1,\frac{\partial\tilde{f}_1}{\partial X_2}\right)+\operatorname{mult}_{\tilde{a}_k}\left(\tilde{f}_1,\tilde{f}_2\right)$$

From the Teissier lemma [3] we infer that

$$\text{mult}_{\tilde{a}_k}\left(\tilde{f}_1, \frac{\partial \tilde{f}_1}{\partial X_2}\right) = M_k + \text{ord}_{\tilde{a}_k} \tilde{f}_1 - 1$$

thus

$$(\sigma - 1)\text{ord}_{\tilde{a}_k} \tilde{f}_1 + \text{mult}_{\tilde{a}_k}\left(\tilde{f}_1, \tilde{J}_f\right) = M_k + \text{mult}_{\tilde{a}_k}\left(\tilde{f}_1, \tilde{f}_2\right) - 1, \quad 1 \le k \le s$$

Summing the above equalities over all points at infinity we have

$$(\sigma - 1)n_1 + \text{mult}_\infty\left(f_1, J_f\right) = \sum_{k=1}^{s} M_k + \text{mult}_\infty\left(f_1, f_2\right) - s$$

By the Bezout theorem

$$\text{mult}_\infty\left(f_1, J_f\right) = n_1 \deg J_f - q \quad \text{and} \quad \text{mult}_\infty\left(f_1, f_2\right) = n_1 n_2 - p$$

From the above we conclude

$$(n_1 - 3)n_1 = \sum_{k=1}^{s} M_k + q - p - s$$

and

$$(n_1 - 1)(n_1 - 2) = \sum_{k=1}^{s} M_k + q - p - s + 2$$

which gives

$$\frac{1}{2}(n_1 - 1)(n_1 - 2) = \frac{1}{2}\sum_{k=1}^{s} M_k + \frac{1}{2}(q - p - s) + 1 \tag{9}$$

Subtracting (6) from (9) we have

$$\frac{1}{2}(n_1 - 1)(n_1 - 2) - \sum_{k=1}^{s} \delta_k = \frac{1}{2}\left(q - p - \sum_{k=1}^{s} r_k\right) + 1$$

In the above equality the number on the left hand side is non-negative integer not less than g_1 [3]. Thus

$$q - p - \sum_{k=1}^{s} r_k \ge 2g_1 - 2$$

which proves the fact.

References

[1] Biernat G., The residue at infinity and Bezout's theorem, Prace Naukowe IMiI, Częstochowa 2002, 25-27.

[2] Krasiński T., Poziomice wielomianów dwóch zmiennych a hipoteza jakobianowa, Acta Universitatis Lodziensis, Wyd. UŁ, Łódź 1991.

[3] Płoski A., O niezmiennikach osobliwości krzywych analitycznych, Materiały VIII Konferencji Szkoleniowej z Teorii Zagadnień Ekstremalnych, Wyd. UŁ, Łódź 1985, 80-93.