SOME REMARKS ON THE TOLERANCE AVERAGED MODEL OF UNIPERIODIC COMPOSITES

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Abstract. The subject of this note are tolerance averaged models. These models can be obtained either by the mode shape functions approach or the basis functions approach [1]. The aim of this contribution is to present a comparison between two simplest versions of such models, i.e. models based on only one shape function. In the case of the basis function approach such model is named in the note as *the heuristic tolerance averaged model* and in the case of the mode shape functions approach as *the tolerance averaged mode shape model*. As a comparison criterion is taken into account the problem of free-vibration of an unbounded two-layered laminate.

1. Introduction

The characteristic feature in the applying of the tolerance averaging technique in the modeling of composites is that in every special problem a certain system of shape functions must be formulated *a priori*. As a rule, in the case of two-layered uniperiodic composites, the saw-like function takes a role of the shape function. This approach has a clear physical motivation in the stationary problems, in which the choice of the saw-like shape function is related to the assumption that the continuous mass distribution can be replaced by the periodic discrete distribution concentrated at the interfaces of the layers. However, such physical motivation may be not realistic in the dynamical problems in which shape functions should represent free vibrations of the periodicity cell. In such models the periodicity cell displacement fluctuation field is usually approximated by the sum of the first N terms of a certain Fourier expansion of periodicity cell vibrations and the shape function system represents the orthogonal basis related to the Fourier expansion. The tolerance averaged models analyzed in this note are related exclusively to the dynamics of laminates made of two linear elastic constituents and are based exclusively on one shape function. The problem is whether one of these models is an acceptable approximation to the second one in the dynamical problems for the composites under consideration.

Let us consider two-component uniperiodic linear-elastic laminate with a period l and denote by a and l-a its layer thicknesses. Moreover let the layers of this laminate be made of the linear elastic materials with mass densities ρ_1, ρ_2 and Young modulus E_1, E_2, respectively. It means that $\rho(\cdot) = \rho_1\chi_1(\cdot) + \rho_2\chi_2(\cdot)$ and

$E(\cdot) = E_1\chi_1(\cdot) + E_2\chi_2(\cdot)$, where $\chi_1(\cdot)$ $\chi_2(\cdot)$ are characteristic functions of sets Ω_1 and Ω_2 related to corresponding materials. Let us denote $\Delta_1(x) = x + \Delta_1$, $\Delta_2(x) = x + \Delta_2$ and $\Delta(x) = x + \Delta$ for $\Delta_1 = (-a/2, a/2)$, $\Delta_2 = (-l/2,-a/2) \cup (a/2,l/2)$, $\Delta = (-l/2, l/2)$. Sets Δ, $\Delta(x)$ will be named as a *basic cell*, and a *periodic cell with the center in x*, respectively. We also denote $\eta_1 = a/l$ and $\eta_2 = 1 - a/l$ and by

$$\langle f\rangle(x) = \frac{1}{l}\int_{-l/2}^{l/2} f(x+y)dy \tag{1}$$

the averaging operator of any integrable function *f*. In this note we restrict considerations to the simplest case of the tolerance averaged model of the composite in which only one shape function is taken into account. This shape function is a certain Δ-periodic continuous function. The averaged value $\langle h\rangle$ of any shape function $h(\cdot)$ vanishes. Without loss of generality, we impose on a shape functions $h(\cdot)$, the additional normalization condition of the form $h(a/2) = \sqrt{3}l$. The model under consideration in this note will be described by the following system of equations:

$$\begin{aligned} &\langle E\rangle u_o'' + \langle Eh'\rangle w' - \langle\rho\rangle \ddot{u}_o = \langle f\rangle \\ &\langle \rho h^2\rangle \ddot{w} + \langle E(h')^2\rangle w + \langle Eh'\rangle u_o' = \langle fh\rangle \end{aligned} \tag{2}$$

in which the averaged displacement field $u_o(\cdot,t)$ and the internal variable field $w(\cdot,t)$ are the basic unknown fields. For the details the reader is referred to [2].
The Δ-periodic shape function $h(\cdot)$ will be treated here as having two meanings. Firstly, it will be identified with the first vibration mode of the periodicity cell. As it will be shown in the problem examined below the choice of one shape function leads not every to acceptable approximations but also on this way it is possible to obtain an exact standing wave-type harmonic solution. Secondly, the shape function $h(\cdot)$ will be identified as the saw-like function; i.e. the simplest piecewise linear basis function chosen between those usually taken into account in the *FEM* method. This approach leads to the heuristic model. We are going to compare two approaches mentioned above.

2. Formulation of the problem

Let us consider the problem of existence of the first mode of Δ-periodic harmonic vibrations of an unbounded two-layered linear elastic laminate. Namely, we are looking for the displacement standing wave-type solutions of the form $u(x,t) = \upsilon(x)\exp(i\omega t)$ to the model equations for the laminate under consideration for a certain Δ-periodic amplitude $\upsilon(\cdot)$. We are to analyze this problem in the framework of three different models.

1) *Exact periodic formulation in the framework of the linear elastodynamics*. In order to formulate this problem we introduce the set of test functions

$$V = \{\upsilon \in H^1_{per}(-l/2, l/2) : \upsilon(-l/2) = \upsilon(l/2) = 0, [\sigma](-a/2) = [\sigma](a/2) = 0\} \quad (3)$$

where $[\sigma](\cdot) \equiv E_1\upsilon'(\cdot)^+ - E_2\upsilon'(\cdot)^-$. Thus the analysis of Δ-periodic harmonic vibrations $u(x,t) = \upsilon(x)\exp(i\omega t)$ leads to the following eigenvalue problem:

Find the smallest eigenvalue ω_0^2 *and the related eigenfunction* $\upsilon_0(\cdot) \in V$ *such that* $\upsilon(y) = \upsilon(-y)$ *for* $y \in R$ *and satisfying the equation*

$$\langle E\upsilon_0'\bar{\upsilon}'\rangle - \omega_0^2\langle\rho\upsilon_0\bar{\upsilon}\rangle = 0 \quad (4)$$

for every test function $\bar{\upsilon}(\cdot) \in V$.

In the framework of the above formulation it is possible to obtain the solution to the above problem. It can be written in the form

$$\upsilon_0(y) = \begin{cases} -\sqrt{3l}\,\dfrac{\sin\omega_0 l(1+2y/l)/2\sqrt{E_2/\rho_2}}{\sin\eta_2\omega_0 l/2\sqrt{E_2/\rho_2}} & for\ \ y \in (-l/2, -a/2) \\ \sqrt{3l}\,\dfrac{\sin\omega_0 y/\sqrt{E_1/\rho_1}}{\sin\eta_1\omega_0 l/2\sqrt{E_1/\rho_1}} & for\ \ y \in (-a/2, a/2) \\ \sqrt{3l}\,\dfrac{\sin\omega_0 l(1-2y/l)/2\sqrt{E_2/\rho_2}}{\sin\eta_2\omega_0 l/2\sqrt{E_2/\rho_2}} & for\ \ y \in (a/2, l/2) \end{cases} \quad (5)$$

where the smallest vibration frequency ω_0 satisfies the frequency equation

$$\sin(\eta_1\sqrt{\frac{\rho_1}{E_1}} + \eta_2\sqrt{\frac{\rho_2}{E_2}})\frac{\omega l}{2} + \frac{\beta-1}{\beta+1}\sin(\eta_1\sqrt{\frac{\rho_1}{E_1}} - \eta_2\sqrt{\frac{\rho_2}{E_2}})\frac{\omega l}{2} = 0 \quad (6)$$

The subsequent two models, which we are going to analyze are certain approximate models which can be obtained by applying the tolerance averaging technique to governing equations of elastodynamics.

2) *Tolerance averaged models for the linear elastodynamics*. In this case the displacement field $u(\cdot,t)$ is represented by the decomposition $u = u_o + u_1$ of the displacement field onto slowly varying averaged part $u_o = \langle u\rangle$, and almost periodic fluctuation part $u_1 = u\text{-}\langle u\rangle$ of the displacement field [2]. The characteristic feature of the tolerance averaging technique is that the periodic approximation u_x of the fluctuation part u_1 of the displacement field in every periodic cell $\Delta(x)$ is represented by its Galerkin approximation which in the framework of this paper is

fluctuation part $u_1 = u\text{-}\langle u\rangle$ of the displacement field [2]. The characteristic feature of the tolerance averaging technique is that the periodic approximation u_x of the fluctuation part u_1 of the displacement field in every periodic cell $\Delta(x)$ is represented by its Galerkin approximation which in the framework of this paper is given by $u_x(y,t) \cong h(y)w(x,t)$, $y \in \Delta(x)$. The characteristic feature of the tolerance averaged models is that the shape functions must be postulated *a priori* in every special problem. Hence, in the framework of just examined tolerance averaged models the shape function $h(\cdot)$ must be known. In the framework of these models the problem of finding Δ-periodic harmonic vibrations of the form $u(x,t) = \upsilon(x)\exp(i\omega t)$ leads to the analysis of the following representation of basic unknowns:

$$\begin{aligned} u_o(x,t) &= 0 \\ w(x,t) &= \upsilon_h(x)\exp(i\omega_h t) \end{aligned} \tag{7}$$

in which a frequency ω_h and an unknown amplitude $\upsilon_h(\cdot)$ are strictly connected with the choice of the shape function $h(\cdot)$ by setting $\upsilon_h = h$. The representations (7) leads to the formula

$$\omega_h^2 = \frac{\langle E(h')^2\rangle}{\langle \rho h^2\rangle} \tag{8}$$

for the investigated frequency.

In the case of *tolerance averaged mode shape model for the linear elastodynamics*, i.e. if the shape function coincides with the first mode shape function, $h = \upsilon_0$, we obtain the formula for the first higher order frequency of the Δ-periodic unbouded two-layered linear elastic laminate in the form

$$\omega_m^2 = \frac{\langle E(\upsilon_0')^2\rangle}{\langle \rho \upsilon_0^2\rangle} \tag{8m}$$

In the case of *the heuristic tolerance averaged model* the shape function coincides with the Δ-periodic saw-like function, $h = h_s$, where

$$h_s(y) = \begin{cases} -\sqrt{3l}\,(1+2y/l)/\eta_2 & \textit{for} \quad y \in (-l/2, -a/2) \\ 2\sqrt{3l}\,y/\eta_1 & \textit{for} \quad y \in (-a/2, a/2) \\ \sqrt{3l}\,(1-2y/l)/\eta_2 & \textit{for} \quad y \in (a/2, l/2) \end{cases} \tag{8n}$$

Hence, we obtain the formula for the first higher order frequency of the Δ-periodic unbounded two-layered linear elastic laminate in the form

$$\omega_s^2 = \frac{\langle E(h_s')^2 \rangle}{\langle \rho h_s^2 \rangle} \tag{8s}$$

In the next section we are to compare just obtained formulas for the first higher order frequencies of Δ-periodic unbounded two-layered linear elastic laminate.

3. Comparison of the models

Now we shall analyze the interrelation between frequencies ω_o, ω_m and ω_s. Firstly, let us note that Eq.(8*m*) is a simple consequence of Eq.(4). Indeed, if we replace in Eq.(4) test function $\bar{\upsilon}$ by the first Δ-periodic mode vibration υ_0 we obtain exactly Eq.(8*m*). Hence, we have just proved the following lemma.

Lemma 1. *The first higher order frequency ω_m of the Δ-periodic unbounded two-layered linear elastic laminate obtained in the framework of the tolerance averaged mode shape model for the linear elastodynamics coincides with the same frequency ω_0 obtained in the framework of the linear elastodynamics for the standing harmonic wave.*

The next step of our considerations is a certain reformulation of the frequency equation (6). To this end let us introduce into considerations the dimensionless parameter

$$\theta_m = \left(\eta_1 \frac{\rho_1}{E_1} + \eta_2 \frac{\rho_2}{E_2} \right) \frac{\omega_0 l}{2} \tag{9}$$

Moreover, let us introduce the denotations

$$\alpha = \frac{\eta_1}{\eta_2} \sqrt{\frac{E_1 \rho_2}{\rho_1 E_2}}, \quad \beta = \sqrt{\frac{E_1 \rho_1}{\rho_2 E_2}} \tag{10}$$

for two new material parameters. By virtue of Eqs.(9), (10) the frequency equation (6) can be rewritten to the form

$$\theta_m = \pi - \arcsin\left(\frac{1-\beta}{1+\beta} \sin \frac{\alpha - 1}{\alpha + 1} \theta_m \right) \tag{11}$$

It must be emphasized that the above form of the frequency equation can be treated as the definition of the function $\theta_m = f_\theta(\alpha, \beta)$ which to every pair (α,β) of material parameters defined in (10) assigns the dimensionless parameter θ_m defined by (9), but simultaneously it must be remembered that θ_m is for every laminate uniquely determined by Eq.(9) as a smallest positive solution ω_0 to the frequency equation (6). Hence, we have just shown the subsequent lemma.

Lemma 2. *Dimensionless parameter* θ_m *and the ratio* ω_s/ω_0 *are uniquely determined by material parameters* α *and* β*. Namely* $\theta_m = f_\theta(\alpha,\beta)$*, where* f_θ *is given by (11) and* $\omega_s/\omega_0 = f_\omega(\alpha,\beta)$*, where*

$$f_\omega(\alpha,\beta) = \frac{3}{\theta_m^2}(1+\alpha)(1+\beta)^2 \big/ (1+\alpha^2\beta) \tag{12}$$

All pairs (α,β) for which the heuristic model based on the saw-like function leads to the frequency ω_s, i.e. which satisfies a formula $\omega_s/\omega_0 = f_\omega(\alpha,\beta) = 1$ together with formulas $\omega_s/\omega_0 = f_\omega(\alpha,\beta) = 1.05$ and $\omega_s/\omega_0 = f_\omega(\alpha,\beta) = 0.95$ are given in Figure 1.

Assuming the accuracy of calculations not larger then 5% it easy to observe that in the band of pairs (α, β) the heuristic model leads to the admissible solution to the problem mentioned above.

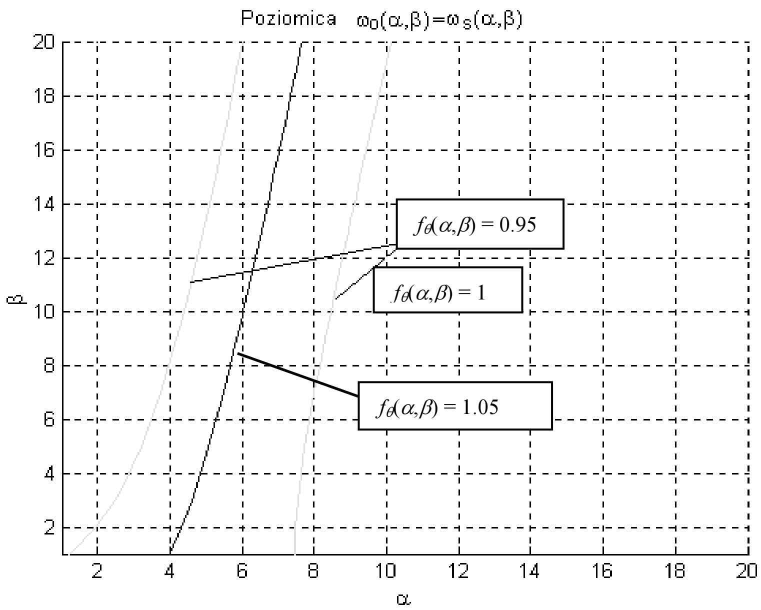

Fig. 1. Diagrams of relations $f_\omega(\alpha,\beta) = 0.95$, $f_\omega(\alpha,\beta) = 1$, $f_\omega(\alpha,\beta) = 1.05$

Conclusions

Summarizing the above considerations we can formulate the following conclusions:

1. In the problem of Δ-periodic standing wave harmonic vibrations of an unbounded two-layered linear elastic laminate the dimensionless parameter θ_m and

the fraction ω_s / ω_0 are uniquely determined by a material parameters α and β defined by Eq. (10).

2. The heuristic model based on the saw-likle function (which leads to acceptable results for stationary problems) not always leads to the acceptable approximate solutions; in many cases it may leads to large errors in the modelling of elasto-dynamics problems.
3. In cases in which the heuristic model leads to considerably errors and hence cannot be applied it is justified to look for the possible modifications of the heuristic model leading to the replacement the saw-like function by more complicated shape functions.

References

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[2] Woźniak C., Wierzbicki E., Averaging techniques in thermomechanics of composite solids, Wydawnictwo Politechniki Częstochowskiej, Częstochowa 2000.