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ANALYSIS OF OPEN QUEUEING NETWORKS USING METHOD OF GENERATING FUNCTIONS

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Abstract. This paper provides the method of generating functions using for calculating the time-dependent state probabilities for open queuing networks in transient regime, when the network works in conditions of peak demand.

Introduction

The queuing networks (QN) can be used as stochastic models of various computer systems and networks (CSN), various objects in economy, production, insurance, medicine and other fields [1-3]. The process of QN analyzing is equal to calculating the state probabilities; because using them we can find all QN characteristics of interest. It should be noticed that various real objects, including CSN, have time-dependent parameters [2, 4]. Having projecting such objects it's often necessary to model their current behavior, to find their time-dependent characteristics. That is why we have to know the non-stationary state probabilities of QN, being used as their models. The exact results for state probabilities of QN in transient regime are obtained in rare cases only [5-7], because of large dimension of the equation set they satisfy. The method of diffusive approximation is used for calculating the state probabilities of QN, working in case of peaking operation [8, 10]. In papers [1, 11, 12] the method of successive approximations, combined with numeric series methods, is developed for finding non-stationary state probabilities of Markovian QN with multi-type messages and multi-line queuing systems (QS), it is also described the application of this method for various closed networks. The open QN, working in peak demand conditions, is analyzed in this paper. We applied the method of generating functions for calculating the state probabilities in transient regime, formerly used for finding queuing systems stationary state probabilities mainly [13].

1. The set of different-differential equations for QN state probabilities

Let us examine the open QN, consisting of n QS, each of them has m_i same servicing lines (see Fig. 1). The duration of message processing in every line of QS S_i has the exponential distribution with service rate equal to $\mu_i, i = \overline{1,n}$. A Poisson message flow with rate λ comes into the network and any incoming message goes to the QS S_i with probability p_{0i}, $i=\overline{1,n-1}$, $\sum_{i=1}^{n-1} p_{0i}=1$ and moves to QS S_n after processing. Taking into consideration the results of Poisson flow sifting, obtained in [13] we can suggest that the incoming message flow for each QS S_i is also a Poisson flow with rate $\lambda_i=\lambda p_{0i}$, $i=\overline{1,n-1}$. The discipline of message processing in systems - FIFO, each of systems S_i, $i=\overline{1,n}$ has the infinite number of places in queue.

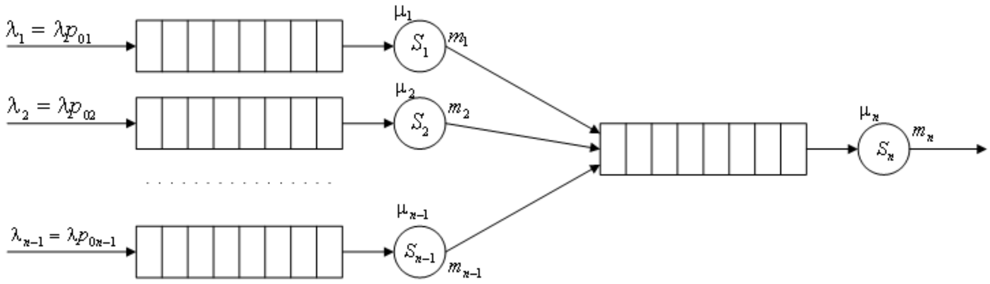

Fig. 1. The open queuing network

The state of QN in moment of time t is defined as the following vector $k(t)=(k,t)=(k_1,k_2,...,k_n,t)$, where k_i is the number of messages in S_i, $i=\overline{1,n}$. Let's suppose that I_i is a vector with zero components, except i-th component, which is equal to 1.

Owing to the fact of exponential distribution of message processing duration, the random process $k(t)$ = (k,t) is a Markovian process with enumerable set of states. There are five possible transitions to state (k,t) during the period of time Δt:

a) from state $(k+I_i-I_n,t)$ with probability

$$\mu_i \cdot u(k_n)\cdot \min(m_i,k_i+1)\Delta t+o(\Delta t), \qquad i=\overline{1,n-1}$$

where $u(x)=\begin{cases}1, x>0\\ 0, x\le 0\end{cases}$

b) from state $(k-I_i,t)$ with probability $\lambda p_{0i}\cdot u(k_i)\cdot \Delta t+o(\Delta t), i=\overline{1,n-1}$

c) from state $(k+I_n,t)$ with probability $\mu_n \cdot \min(m_n,k_n+1)\Delta t+o(\Delta t)$

d) from state (k,t) with probability

$$1-\left[\sum_{i=1}^{n}\mu_i\cdot \min(m_i,k_i+1)+\lambda\right]\Delta t+o(\Delta t)$$

e) from all other states with probability $o(\Delta t)$.

Then, using the formula of total probability, we can suggest

$$P(k,t+\Delta t)=\sum_{i-1}^{n-1}P(k+I_i-I_n,t)\mu_i u(k_n)\min(m_i,k_i+1)\Delta t+\sum_{i=1}^{n-1}P(k-I_i,t)\lambda p_{0i}u(k_i)\Delta t+$$

$$+P(k+I_n,t)\mu_n\min(m_n,k_n+1)\Delta t+P(k,t)\left\{1-\left[\sum_{i=1}^{n}\mu_i\min(m_i,k_i+1)+\lambda\right]\Delta t\right\}+o(\Delta t)$$

While dividing both parts of this statement into Δt and proceeding to the limit when $\Delta t\to 0$, we'll get the set of different-differential equations for QN state probabilities:

$$\frac{dP(k,t)}{dt}=-\left\{\lambda+\sum_{i=1}^{n}\mu_i\min(m_i,k_i+1)\right\}P(k,t)+\lambda\sum_{i=1}^{n-1}P(k-I_i,t)p_{0i}u(k_i)+$$

$$+\sum_{i=1}^{n-1}P(k+I_i-I_n,t)\mu_i u(k_n)\min(m_i,k_i+1)+\mu_n\min(m_n,k_n+1)P(k+I_n,t)\quad(1)$$

2. The state probabilities of QN in case n = 3

Let n = 3 and all queuing systems of QN have one servicing line only, i.e. $m_i=1, i=\overline{1,n},$ the equations set (1) will look as follows:

$$\frac{dP(k,t)}{dt}=-\left\{\lambda+\sum_{i=1}^{3}\mu_i\right\}P(k,t)+\lambda\sum_{i=1}^{2}P(k-I_i,t)p_{0i}u(k_i)+$$
$$+\sum_{i=1}^{2}P(k+I_i-I_n,t)\mu_i u(k_3)+\mu_3P(k+I_3,t)$$

Let us define the many-dimensional generating function. Assuming $z=(z_1,z_2,z_3)$ we'll get the following:

$$P(z,t)=\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}P(k_1=l,k_2=m,k_3=r,t)z_1^l z_2^m z_3^r=$$
$$=\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}P(l,m,r,t)z_1^l z_2^m z_3^r\quad(2)$$

Multiplying each of the equations (1) by $z_1^l z_2^m z_3^r$ and summing over all possible values of l,m,r we'll obtain the statement

$$\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\frac{dP(k,t)}{dt}z_1^l z_2^m z_3^r = -\left\{\lambda + \sum_{i=1}^{3}\mu_i\right\}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}P(k,t)z_1^l z_2^m z_3^r +$$

$$+\lambda\sum_{i=1}^{2}p_{0i}u(k_i)\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}P(k-I_i,t)z_1^l z_2^m z_3^r + \sum_{i=1}^{2}\mu_i u(k_3)\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}P(k+I_i-I_3,t)z_1^l z_2^m z_3^r +$$

$$+\mu_3\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}P(k+I_3,t)z_1^l z_2^m z_3^r$$

Further, we'll examine several right-hand sums of this statement separately. Let $\sum_1(x,t) = \sum_{i=1}^{2}p_{0i}u(k_i)\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}P(k-I_i,t)z_1^l z_2^m z_3^r$. Then

$$\sum_1(x,t) = p_{01}u(k_1)\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}P(k-I_1,t)z_1^l z_2^m z_3^r + p_{02}u(k_2)\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}P(k-I_2,t)z_1^l z_2^m z_3^r =$$

$$= p_{01}z_1\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}P(l-1,m,r,t)z_1^{l-1}z_2^m z_3^r + p_{02}z_2\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}P(l,m-1,r,t)z_1^l z_2^{m-1}z_3^r =$$

$$= p_{01}z_1\sum_{s=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}P(s,m,r,t)z_1^s z_2^m z_3^r + p_{02}z_2\sum_{l=0}^{\infty}\sum_{u=0}^{\infty}\sum_{r=0}^{\infty}P(l,u,r,t)z_1^l z_2^u z_3^r$$

$$= (p_{01}z_1 + p_{02}z_2)P(z,t)$$

For sum $\sum_2(z,t) = \sum_{i=1}^{2}\mu_i u(k_3)\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}P(k+I_i-I_3,t)z_1^l z_2^m z_3^r$ we'll obtain:

$$\sum_2(z,t) = \mu_1 u(k_3)\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}P(k+I_1-I_3,t)z_1^l z_2^m z_3^r + \mu_2 u(k_3)\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}P(k+I_2-I_3,t)z_1^l z_2^m z_3^r =$$

$$= \mu_1\frac{z_3}{z_1}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=1}^{\infty}P(l+1,m,r-1,t)z_1^{l+1}z_2^m z_3^{r-1} + \mu_2\frac{z_3}{z_2}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=1}^{\infty}P(l,m+1,r-1,t)z_1^l z_2^{m+1}z_3^{r-1} =$$

$$= \mu_1\frac{z_3}{z_1}\sum_{l=1}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}P(l,m,r,t)z_1^l z_2^m z_3^r + \mu_2\frac{z_3}{z_2}\sum_{l=0}^{\infty}\sum_{m=1}^{\infty}\sum_{r=0}^{\infty}P(l,m,r,t)z_1^l z_2^m z_3^r =$$

$$= z_3\left(\frac{\mu_1}{z_1} + \frac{\mu_2}{z_2}\right)P(z,t) - \mu_1\frac{z_3}{z_1}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}P(0,m,r,t)z_2^m z_3^r - \mu_2\frac{z_3}{z_3}\sum_{l=0}^{\infty}\sum_{u=0}^{\infty}P(l,0,r,t)z_1^l z_3^r$$

And, finally, the last sum $\sum_3(z,t)=\mu_3\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}P(l,m,r+1,t)z_1^l z_2^m z_3^r$ can be simplified as follows:

$$\sum_2(z,t)=\frac{\mu_3}{z_3}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}P(l,m,r+1,t)z_1^l z_2^m z_3^{r+1}=\frac{\mu_3}{z_3}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=1}^{\infty}P(l,m,r,t)z_1^l z_2^m z_3^r=$$

$$=\frac{\mu_3}{z_3}P(z,t)-\frac{\mu_3}{z_3}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}P(l,m,0,t)z_1^l z_2^m$$

Thus, considering notation (2) for generating function we'll obtain heterogeneous linear differential equation

$$\frac{dP(z,t)}{dt}=-\left\{\lambda+\sum_{i=1}^{3}\mu_i-\lambda p_{01}z_1-\lambda p_{02}z_2-z_3\left(\frac{\mu_1}{z_1}+\frac{\mu_2}{z_2}\right)-\frac{\mu_3}{z_3}\right\}P(z,t)-$$

$$-\mu_1\frac{z_3}{z_1}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}P(0,m,r,t)z_2^m z_3^r-\mu_2\frac{z_3}{z_2}\sum_{l=0}^{\infty}\sum_{r=0}^{\infty}P(l,0,r,t)z_1^l z_3^r-$$

$$-\frac{\mu_3}{z_3}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}P(l,m,0,t)z_1^l z_2^m \tag{3}$$

Let us suppose, that all QS of QN work in peak demand conditions, i.e. $k_i(t)>0,\forall t>0,i=\overline{1,n}$ (that means that there is at least one message being serviced in any QS). In that case the last three items in equation (3) are equal to zero, and therefore it becomes heterogeneous

$$\frac{dP(z,t)}{dt}=-\left\{\lambda+\sum_{i=1}^{3}\mu_i-\lambda p_{01}z_1-\lambda p_{02}z_2-z_3\left(\frac{\mu_1}{z_1}+\frac{\mu_2}{z_2}\right)-\frac{\mu_3}{z_3}\right\}P(z,t) \tag{4}$$

Let's consider, that at start time the concerned QN is empty, i.e. $P(0,0,0,0) = 1$. Then the starting condition for equation (3) will look like $P(z,0) = 1$. Thus, we obtained the heterogeneous linear differential equation (4) for generating function with time-independent coefficients. Having solved this equation and considered it's starting condition, we'll obtain

$$P(z,t)=\exp\left[-\left\{\lambda+\sum_{i=1}^{3}\mu_i-\lambda p_{01}z_1-\lambda p_{02}z_2-z_3\frac{\mu_1}{z_1}-z_3\frac{\mu_2}{z_2}-\frac{\mu_3}{z_3}\right\}t\right] \tag{5}$$

Based upon generating function determination (2), the probability $P(k,t)$, $k=(k_1,k_2,k_3)$ is equal to coefficient of $z_1^{k_1}z_2^{k_2}z_3^{k_3}$ in expansion of $P(z,t)$.

Let us reorganize statement (5) for more convenient form. If it is denoted $a_0(t) = \exp\left\{ -(\lambda + \sum_{i=1}^{3} \mu_i)t \right\}$, then

$$P(z,t) = a_0(t)\exp(\lambda p_{01} z_1 t)\exp(\lambda p_{02} z_2 t)\exp(z_3 \frac{\mu_1}{z_1} t)\exp(z_3 \frac{\mu_2}{z_2} t)\exp(\frac{\mu_3}{z_3} t) =$$

$$= a_0(t)\sum_{i=0}^{\infty}\frac{(\lambda p_{01} z_1 t)^i}{i!}\sum_{j=0}^{\infty}\frac{(\lambda p_{02} z_2 t)^j}{j!}\sum_{l=0}^{\infty}\frac{(\mu_1 z_1^{-1} z_3 t)^l}{l!}\sum_{s=0}^{\infty}\frac{(\mu_2 z_2^{-1} z_3 t)^s}{s!}\sum_{q=0}^{\infty}\frac{(\mu_3 z_3^{-1} t)^q}{q!} =$$

$$= a_0(t)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\sum_{u=0}^{\infty}\frac{\lambda^{i+j} p_{01}^i p_{02}^j \mu_1^l \mu_2^s \mu_3^q}{i!\,j!\,l!\,s!\,q!} t^{i+j+l+s+q} z_1^{i-l} z_2^{j-s} z_3^{l+s-q}$$

Example 1. Let's calculate the probability of state $P(1,1,1,t)$, which is equal to coefficient of $z_1 z_2 z_3$ in expansion of $P(x,t)$. At that, the powers of $z_1 z_2 z_3$ satisfy the following statements:

$$\begin{cases} i - l = 1 \\ j - s = 1 \\ l + s - q = 1 \end{cases} \qquad \text{i.e.} \qquad \begin{cases} l = i - 1 \\ s = j - 1 \\ q = i + j - 3 \\ i, j > 0 \end{cases}$$

Thus

$$P(1,1,1,t) = e^{-(\lambda + \sum_{i=1}^{3}\mu_i)t}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\lambda^{i+j} p_{01}^j p_{02}^j \mu_1^{i-1}\mu_2^{j-1}\mu_3^{i+j-3}}{i!\,j!\,(i-1)!\,(j-1)!\,(i+j-3)!} t^{3i+3j-5}$$

Assumed that $p_{01} = 0.3,\ p_{02} = 0.7, \mu_1 = 1,\ \mu_2 = 2,\ \mu_3 = 10,\ \lambda = 1$ we'll get the following graph (see Fig. 2 below) for time-dependent probability $P(1,1,1,t)$.

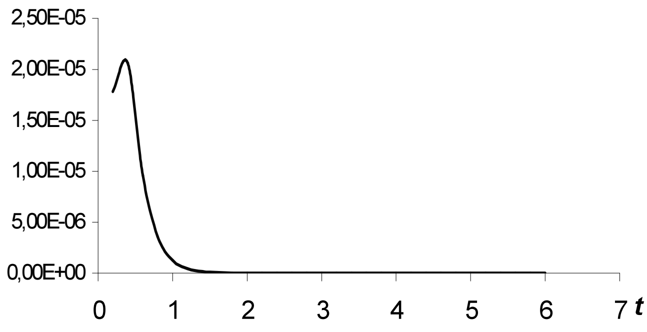

Fig. 2. The graph of $P(1,1,1,t)$

Example 2. Let's find $P(2,3,5,t)$ now, which is equal to coefficient of $z_1^2 z_2^3 z_3^5$ in expansion of $P(z,t)$. The statements for powers will look like:

$$\begin{cases} i-l=2 \\ j-s=3 \\ l+s-q=5 \end{cases} \quad \text{i.e.} \quad \begin{cases} l=i-2 \\ s=j-3 \\ q=i+j-10 \\ i,j>0 \end{cases}$$

Thus

$$P(2,3,5,t)=e^{-(\lambda+\sum_{i=1}^{3}\mu_i)t}\cdot\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\lambda^{i+j}p_{01}^{j}p_{02}^{j}\mu_1^{i-2}\mu_2^{j-3}\mu_3^{i+j-10}}{i!\,j!(i-2)!(j-3)!(i+j-10)!}t^{3i+3j-15}$$

Assumed that $p_{01}=0.3,\ p_{02}=0.7, \mu_1=1,\ \mu_2=2,\ \mu_3=10,\ \lambda=1$ we'll get the following graph (see Fig. 3 below) for time-dependent probability $P(2,3,5,t)$.

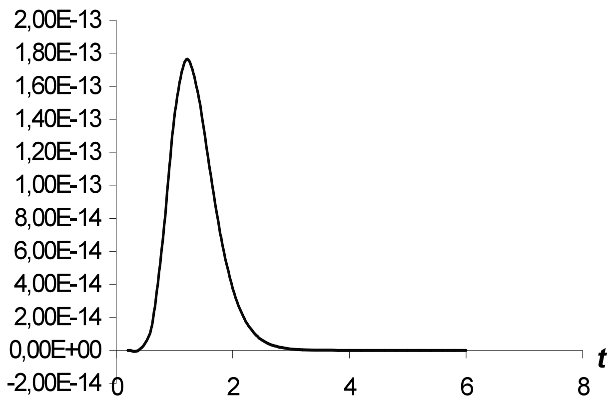

Fig. 3. The graph of $P(2,3,5,t)$

3. The state probabilities of QN in case of arbitrary *n*

Let us summarize our results for case of arbitrary number of QS, but like in previous case, it's supposed that all of them have one servicing line only. Then the equations set for state probabilities (1) will look like:

$$\frac{dP(k,t)}{dt}=-\left\{\lambda+\sum_{i=1}^{n}\mu_i\right\}P(k,t)+\lambda\sum_{i=1}^{n-1}P(k-I_i,t)p_{0i}u(k_i)+$$
$$+\sum_{i=1}^{n-1}P(k+I_i-I_n,t)\mu_i u(k_n)+\mu_n P(k+I_n,t)$$

Let us denote that all systems of QN work in peak demand conditions as before, i.e. $k_i(t) > 0, \forall t > 0, i = \overline{1,n}.$ Let us define the n-dimensional generating function:

$$P(z,t) = \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}...\sum_{i_n=0}^{\infty} P(i_1, i_2,..,i_n,t)\prod_{k=1}^{n} z_k^{i_k}$$

Performing research similar to case $n = 3$ we'll get the heterogeneous linear differential equation for course-of-value function

$$\frac{dP(z,t)}{dt} = -\left\{\lambda + \sum_{i=1}^{n}\mu_i - \lambda\sum_{i=1}^{n-1} p_{0i}z_i - z_n\sum_{i=1}^{n-1}\frac{\mu_i}{z_i} - \frac{\mu_n}{z_n}\right\}P(z,t)$$

Considering the starting condition $P(z,0) = 1$ the solution if this equation will be the following function

$$P(z,t) = \exp\left[-\left\{\lambda + \sum_{i=1}^{n}\mu_i - \lambda\sum_{i=1}^{n-1} p_{0i}z_i - z_n\sum_{i=1}^{n-1}\frac{\mu_i}{z_i} - \frac{\mu_n}{z_n}\right\}t\right] \tag{6}$$

Let us reorganize statement (6) for more convenient form

$$P(z,t) = \exp\left[-\left\{\lambda + \sum_{i=1}^{n}\mu_i\right\}\cdot t\right]\exp(\lambda\sum_{i=1}^{n-1} p_{0i}z_i t)\exp(z_n\sum_{i=1}^{n-1}\frac{\mu_i}{z_i}t)\exp(\frac{\mu_n}{z_n}t) =$$

$$= a_0(t)a_1(z,t)a_2(z,t)a_3(z,t)$$

where:

$$a_0(t) = \exp\left[-\left\{\lambda + \sum_{i=1}^{n}\mu_i\right\}\cdot t\right]$$

$$a_1(z,t) = \exp(\lambda\sum_{i=1}^{n-1} p_{0i}z_i t) = \exp(\sum_{i=1}^{n-1}\lambda p_{0i}z_i t) = \prod_{i=1}^{n-1}\exp(\lambda p_{0i}z_i t) = \prod_{i=1}^{n-1}\sum_{l_i=0}^{\infty}\frac{(\lambda p_{0i}z_i t)^{l_i}}{l_i!} =$$

$$= \sum_{l_1=0}^{\infty}\frac{(\lambda p_{01}z_1 t)^{l_1}}{l_1!}\sum_{l_2=0}^{\infty}\frac{(\lambda p_{02}z_2 t)^{l_2}}{l_2!}\cdot...\cdot\sum_{l_{n-1}=0}^{\infty}\frac{(\lambda p_{0n-1}z_{n-1}t)^{l_{n-1}}}{l_{n-1}!} =$$

$$= \sum_{l_1=0}^{\infty}\sum_{l_2=0}^{\infty}...\sum_{l_{n-1}=0}^{\infty}\frac{(\lambda p_{01}z_1 t)^{l_1}}{l_1!}\frac{(\lambda p_{02}z_2 t)^{l_2}}{l_2!}\cdot...\cdot\frac{(\lambda p_{0n-1}z_{n-1}t)^{l_{n-1}}}{l_{n-1}!} =$$

$$= \sum_{l_1=0}^{\infty}\sum_{l_2=0}^{\infty}...\sum_{l_{n-1}=0}^{\infty}\frac{\lambda^{l_1+l_2+...+l_{n-1}}t^{l_1+l_2+...+l_{n-1}}}{l_1!l_2!\cdot...\cdot l_{n-1}!}p_{01}^{l_1}p_{02}^{l_2}\cdot...\cdot p_{0n-1}^{l_{n-1}}z_1^{l_1}z_2^{l_2}\cdot...\cdot z_{n-1}^{l_{n-1}}$$

$$a_2(z,t)=\exp(z_n\sum_{i=1}^{n-1}\frac{\mu_i}{z_i}t)=\prod_{i=1}^{n-1}\exp(z_n z_i^{-1}\mu_i t)=\prod_{i=1}^{n-1}\sum_{m_i=0}^{\infty}\frac{(z_n z_i^{-1}\mu_i t)^{m_i}}{m_i!}=$$

$$=\sum_{m_1=0}^{\infty}\frac{(z_n z_1^{-1}\mu_1 t)^{m_1}}{m_1!}\sum_{m_2=0}^{\infty}\frac{(z_n z_2^{-1}\mu_2 t)^{m_2}}{m_2!}\cdot\ldots\cdot\sum_{m_{n-1}=0}^{\infty}\frac{(z_n z_{n-1}^{-1}\mu_{n-1} t)^{m_{n-1}}}{m_{n-1}!}=$$

$$=\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\ldots\sum_{m_{n-1}}^{\infty}\frac{z_n^{m_1+m_2+\ldots+m_{n-1}}t^{m_1+m_2+\ldots+m_{n-1}}}{m_1!m_2!\cdot\ldots\cdot m_{n-1}!}\mu_1^{m_1}\mu_2^{m_2}\cdot\ldots\cdot\mu_{n-1}^{m_{n-1}}\cdot z_1^{-m_1}z_2^{-m_2}\cdot\ldots\cdot z_{n-1}^{-m_{n-1}}$$

$$a_3(z,t)=\exp(\frac{\mu_n}{z_n}t)=\sum_{q=0}^{\infty}\frac{\mu_n^q z_n^{-q}t^q}{q!}$$

Thus

$$P(z,t)=a_0(t)\sum_{l_1=0}^{\infty}\sum_{l_2=0}^{\infty}\ldots\sum_{l_{n-1}=0}^{\infty}\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\ldots\sum_{m_{n-1}=0}^{\infty}\sum_{q=0}^{\infty}\frac{\lambda^{l_1+l_2+\ldots+l_{n-1}}p_{01}^{l_1}p_{02}^{l_2}\cdot\ldots\cdot p_{0n-1}^{l_{n-1}}\mu_1^{m_1}\mu_2^{m_2}\cdot\ldots\cdot\mu_{n-1}^{m_{n-1}}}{l_1!l_2!\cdot\ldots\cdot l_{n-1}!m_1!m_2!\cdot\ldots\cdot m_{n-1}!}\times$$

$$\times z_1^{l_1-m_1}z_2^{l_2-m_2}\cdot\ldots\cdot z_{n-1}^{l_{n-1}-m_{n-1}}z_n^{m_1+m_2+\ldots+m_{n-1}-q}\cdot t^{l_1+l_2+\ldots+l_{n-1}+m_1+m_2+\ldots+m_{n-1}+q}$$

Example 3. Let's find $P(1,1,\ldots,1,t)$ now, which is equal to coefficient of $\prod_{i=1}^{n} z_i$ in expansion of $P(z,t)$. The statements for powers will look like:

$$\begin{cases} l_1-m_1=1\\ l_2-m_2=1\\ \quad\ldots\ldots\ldots\\ l_{n-1}-m_{n-1}=1\\ m_1+m_2+\ldots+m_{n-1}-q=1\end{cases}$$

Having found variables m_i from the first $n-1$ statements of the equations set above and placing them under n-th equation, we'll obtain the following:

$$\begin{cases} m_i=l_i-1\\ q=\sum_{i=1}^{n-1}l_i-n\end{cases}$$

Then

$$P(1,1,\ldots,1,t)=$$

$$\sum_{l_1=0}^{\infty}\sum_{l_2=0}^{\infty}\ldots\sum_{l_{n-1}=0}^{\infty}\frac{\lambda^{l_1+l_2+\ldots+l_{n-1}}\cdot p_{01}^{l_1}p_{02}^{l_2}\cdot\ldots\cdot p_{0n-1}^{l_{n-1}}\cdot\mu_1^{l_1-1}\mu_2^{l_2-1}\cdot\ldots\cdot\mu_{n-1}^{l_{n-1}-1}\mu_n^{l_1+l_2+\ldots+l_{n-1}-n}}{l_1!l_2!\cdot\ldots\cdot l_{n-1}!(l_1-1)!(l_2-1)!\cdot\ldots\cdot(l_{n-1}-1)!(l_1+l_2+\ldots+l_{n-1}-n)!}\times$$

$$\times t^{3l_1+3l_2+\ldots+3l_{n-1}-2n+1}$$

Example 4. If we are interested in calculating the probability of state $P(1,1,1,...,3,t)$, then we have to find the coefficient of $z_1 z_2 \cdot ... \cdot z_n^3$ in expansion of $P(z,t)$. The statements for powers will look like:

$$\begin{cases} l_1 - m_1 = 1 \\ l_2 - m_2 = 1 \\ \quad \dots\dots\dots \\ l_{n-1} - m_{n-1} = 1 \\ m_1 + m_2 + ... + m_{n-1} - q = 3 \end{cases} \qquad \text{i.e.} \begin{cases} m_i = l_i - 1 \\ q = \sum_{i=1}^{n-1} l_i - n - 2 \end{cases}$$

Thus,

$$P(1,1,...,1,t) =$$
$$\sum_{l_1=0}^{\infty}\sum_{l_2=0}^{\infty}...\sum_{l_{n-1}=0}^{\infty} \frac{\lambda^{l_1+l_2+...+l_{n-1}} \cdot p_{01}^{l_1} p_{02}^{l_2} \cdot ... \cdot p_{0n-1}^{l_{n-1}} \cdot \mu_1^{l_1-1} \mu_2^{l_2-1} \cdot ... \cdot \mu_{n-1}^{l_n-1} \mu_n^{l_1+l_2+..+l_{n-1}-n-2}}{l_1! l_2! \cdot ... \cdot l_{n-1}! (l_1-1)!(l_2-1)! \cdot ... \cdot (l_{n-1}-1)!(l_1+l_2+...+l_{n-1}-n-2)!} \times$$
$$\times t^{3l_1+3l_2+...+3l_{n-1}-2n-3}$$

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