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THERMOMECHANICAL RESPONSES OF FUNCTIONALLY GRADED CYLINDERS

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Abstract. In this study, thermal and mechanical stresses in hollow thick-walled functionally graded (FG) cylinders is presented under the convection boundary condition. The convective external condition and constant internal temperature in hollow cylinders are investigated. Inhomogeneous material properties produce irregular and two-point linear boundary value problems that are solved numerically by the pseudospectral Chebyshev method. The displacement and thermal stress distributions are examined for two different material couples under particular boundary conditions that are similar to their real engineering applications. Results have demonstrated that the pseudospectral Chebyshev method has low computation costs, high accuracy and ease of implementation and can be easily customized to such engineering problems.

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Nomenclature

a	inner radius	σ_θ	circumferential stress
b	outer radius	σ_z	axial stress
k	thermal conductivity	P	pressure
α	thermal expansion coefficient	T	temperature distribution
h	thermal convection coefficient	T_∞	ambient temperature
D	Chebyshev differentiation matrix	T_w	wall temperature
E	Young's modulus	Q	internal heat generation
q_0	initial heat generation	ε_r	radial strain
r	radial coordinate	ε_θ	circumferential strain
u	radial displacement	λ, μ	Lame coefficients
σ_r	radial stress	m_j	inhomogeneity parameters (j = 1, 2, 3)

1. Introduction

Functionally graded materials (FGMs) are innovative composite materials whose thermal and mechanical properties vary smoothly from one surface to the next. They offers many benefits, including better thermal characteristics, increased material strength, improved residual stress distribution, high fracture toughness, and relatively low in plane and transverse stresses. The FGM thick-walled cylinder is widely utilized in many industries for delivering and reserving fluids under pressure and temperature loads, including power/chemical plants, aerospace, biomedical, petroleum, and others. FGMs have been the subject of a substantial amount of research [1].

In the past several decades, functionally graded materials have also played an important role in the design of cylindrical structures. Many of the studies deal with temperature, displacement and stress distribution of functionally graded hollow cylinders with different solution procedures. The solution methods used in the literature can be listed as: the perturbation method [2], the direct method [3], a novel limiting approach [4], generalized Bessel function and Fourier integral [5], Fredholm integral equation [6], a tolerance averaging approach [7], Differential quadrature method [8], representative volume element [9], the multilayer semi-analytical method [10], complementary functions method [11], the direct method and finite element method [12], an approximate method for a cylinder divided into N layers [13], the non-linear shooting method and the Runge-Kutta fourth-order algorithm [14].

This paper deals with thick-walled hollow cylinders that are graded with two different material couples. Material distribution is assumed to vary an exponential function in the radial direction. Systems of linear ordinary differential equations with variable coefficients have been solved numerically with the pseudospectral Chebyshev method. Differentiation matrices that signify the approximations at grid points play an important role in the implementation of spectral collocation methods [15]. The constructing procedure of Chebyshev differentiation matrices with the help of Chebyshev points (grid points) is found in Fornberg [16] and the implementation for the numerical solution of a convection-diffusion problem in Bazan [17]. The method is used to generate results for the temperature, displacement, and thermal stress distributions.

2. Mathematical formulation of the problem

Temperature, displacement and stress analysis of a one-dimensional axisymmetric functionally graded cylindrical body is considered. A thick-walled hollow cylinder subjected to steady-state thermal and mechanic loads is investigated. The modulus of elasticity, heat conduction coefficient, the linear thermal expansion coefficient and Poisson's ratio of the body are assumed to be graded exponentially in the radial direction as follows:

$$E(r) = E_m\, e^{\frac{m_1(r-a)}{b-a}},\;\; k(r) = k_m\, e^{\frac{m_2(r-a)}{b-a}},\;\; \alpha(r) = \alpha_m\, e^{\frac{m_3(r-a)}{b-a}},$$
$$\nu(r) = \nu_m\, e^{\frac{m_4(r-a)}{b-a}} \tag{1}$$

Here, subscript m, m_j, $j = 1, 2, 3, 4$, r, a and b represent metal constituents, inhomogeneity parameters, radial coordinates, inner and outer radius, respectively.

Temperature distribution of the FG cylinder in a steady-state condition with internal heat generation is described by the axisymmetric heat conduction equation [18] as follows:

$$\frac{1}{r}\,[r\,k(r)\,T'(r)]' + Q = 0 \tag{2}$$

where the prime refers to the derivative with respect to r, and $T(r)$ is the temperature in the radial direction. Under the assumption of no internal heat generation, substituting the exponential form of the heat conduction coefficient (1) into Eq. (2) renders the heat conduction equation in the following form:

$$T'' + \left(\frac{m_2}{b-a} + \frac{1}{r}\right) T' = 0 \tag{3}$$

with a uniform inner surface temperature and convective outer surface that is exposed to an airstream (h_o = 20 W/m^2·K)

$$T(a) = T_w, \quad \left[k_m \frac{dT}{dr} + h_o(T - T_\infty)\right]_{r=b} = 0 \tag{4}$$

The governing equation for the stress field of the cylindrical body consists of strain-displacement equations

$$\varepsilon_r = \frac{du}{dr}, \quad \varepsilon_\theta = \frac{u}{r} \tag{5}$$

stress-strain-temperature relations

$$\sigma_r = (\lambda(r) + 2\mu(r))\varepsilon_r + \lambda\varepsilon_\theta - (3\lambda(r) + 2\mu(r))\alpha(r)T(r) \tag{6a}$$

$$\sigma_\theta = (\lambda(r) + 2\mu(r))\varepsilon_\theta + \lambda(r)\varepsilon_r - (3\lambda(r) + 2\mu(r))\alpha(r)T(r) \tag{6b}$$

$$\sigma_z = \lambda(r)(\varepsilon_r + \varepsilon_\theta) + -(3\lambda(r) + 2\mu(r))\alpha(r)T(r) \tag{6c}$$

and stress equilibrium equation

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0. \tag{7}$$

Here, σ_r, σ_θ, σ_z represent the radial, hoop and axial stress, and ε_r, ε_θ are the strain tensors. Lame coefficients, $\lambda(r)$ and $\mu(r)$ are related to the modulus of elasticity $E(r)$ and Poisson's ratio ν in the following way:

$$\lambda(r) = \frac{\nu(r)\,E(r)}{[1 + \nu(r)][1 - 2\nu(r)]}, \quad \mu(r) = \frac{E(r)}{2[1 + \nu(r)]} \tag{8}$$

Substituting Eqs. (5)-(6) into the equilibrium Eq. (7), by using Lame coefficients (8) with the exponential form of the modulus of elasticity and the linear thermal expansion coefficient (1) renders the linear non-homogeneous ordinary differential equation (ODE) in terms of radial displacement, u as follows:

$$u'' + P(r)u' + G(r)u = R(r) \tag{9}$$

where

$$P(r) = \frac{\lambda' + 2\mu'}{\lambda + 2\mu} + \frac{1}{r} \tag{10a}$$

$$G(r) = \frac{\lambda'}{\lambda + 2\mu}\frac{1}{r} - \frac{1}{r^2} \tag{10b}$$

$$R(r) = \alpha\, e^{m_3\left(\frac{r-a}{b-a}\right)}\left[\left(\frac{3\lambda' + 2\mu'}{\lambda + 2\mu} + \frac{1+\nu}{1-\nu}\frac{m_3}{b-a}\right)T + \frac{1+\nu}{1-\nu}T'\right]. \tag{10c}$$

3. Numerical resolution of the problem

In the pseudospectral Chebyshev method, a solution is made in the interval determined in the problem. The mesh points that will keep the error to a minimum are selected and the interpolation polynomial at these points is found. In order to obtain high precision solutions by using fewer points in the solution of the problem, Chebyshev Gauss-Lobatto points, which contain a more dense point distribution at the boundary points compared to the midpoints, are preferred. These points are equally spaced on the semicircle in accordance with the equation below.

$$r_j = \cos\left(\frac{j\pi}{n}\right), \quad (j = 0,1,\dots,n) \tag{11}$$

Since they are projected on the horizontal axis, they form a distribution that is dense at the borders and sparse at the midpoints. Thus, high precision solutions can be obtained with fewer grid points.

The pseudospectral Chebyshev Model is utilized to perform the thermal stress analysis of FG hollow cylinders under the convective boundary condition by referring to the study of Trefethen [15], Fornberg [16] and Gottlieb [19] that depends on discretization of the governing equations (3), (9), (12), (15) with respect to the spatial variable using the pseudospectral Chebyshev method. With regard to collocation points, the first order $(n+1)x(n+1)$ Chebyshev differentiation matrix will be obtained and denoted by D. First-order Chebyshev differentiation matrix D provides highly precise approximation to $u'(r_j)$, $T'(r_j)$, $u''(r_j)$, $T''(r_j)$ ..., simply by multiplying the differential matrix with vector data $u'(r_j) = (D\,u)_j$, $T'(r_j) = (D\,T)_j$,

$u''(r_j) = (D^2\, u)_j$, $T''(r_j) = (D^2\, T)_j$ suchlike where $\boldsymbol{u} = [u_0, \dots, u_n]^T$ and $\boldsymbol{T} = [T_0, \dots, T_n]^T$ discrete vectors data at positions r_j.

The computation procedure of the Chebyshev differentiation matrix and codes as an m-file can be found in notable references see e.g. [15], where the collocation points r_j are numbered from right to left and defined in $[-1,1]$. With a small revision, the method can be implemented to any interval.

The detailed implementation of the method is explained in the study of Trefethen [15] and Bazan [17]. Therefore, the linear axisymmetric heat conduction equation for the cylinder (3) is simply converted into a linear system by using the pseudo-spectral Chebyshev collocation method as follows:

$$M_T \boldsymbol{T} = \boldsymbol{0} \tag{12}$$

where

$$M_T = D^2 + \left(\frac{m_2}{b-a} + \frac{1}{r}\right) D. \tag{13}$$

Boundary conditions for temperature (11) are imposed to this linear system (12) by only replacing the first and last row of the system matrix M_T with the appropriate values and the corresponding RHS_T values. Then the nondimensional temperature field can be found by solving the linear system (12) by any decomposition method. After that, the linear non-homogeneous ordinary differential equation (9) in terms of dimensionless radial displacement is converted into a linear system in the following way:

$$M_u \boldsymbol{u} = R(r) \tag{14}$$

where

$$M_u = D^2 + P(r)D + G(r). \tag{15}$$

4. Results and discussion

Cylindrical components can be exposed by a variety of mechanical and thermal effects. In the present study, a hollow thick walled FG cylinder with an inner radius a = 1 m and outer radius b = 1.2 m is examined. It is presumed that the cylinder is exposed to internal pressure $P(a)$ = –50 MPa and constant internal temperature T_w = 50°C. The ambient temperature is set to be T_∞ = 25°C. Materials are listed in Table 1 with their properties. Using the relations in Eq. (1) and the material properties in Table 1, the inhomogeneity parameters m_1, m_2, m_3 and m_4 are calculated so that the inner wall of the cylinder is pure metal and the outer wall is pure ceramic and shown in Table 2.

Table 1. Properties of material couples (*MC*)

	Material	E [GPa]	k [W/mK]	α [10^{-6} K^{-1}]	v
MC_1	Silicon Nitride	348.43	1.209	5.8723	0.24
	Nickel	199.5	90.7	13.30	0.3
MC_2	Mullite	225	5.90	4.7	0.27
	Molybdenum	330	138	4.9	0.3

Table 2. Inhomogeneity parameters for *MC's*

	m_1	m_2	m_3	m_4
MC_1	0.5576	–4.3177	–0.8175	–0.2231
MC_2	–0.3830	–3.1523	–0.0417	–0.1054

As an initial step to the analysis, grid independence tests are conducted for the cylinder and presented in Table 3. It is noted that the spectral procedure is capable of attaining 6, 8, 10 digit precision by picking 11, 13, 15 collocation points, respectively. Therefore, 11 (N = 10 interval) collecting points are used in all analyses conducted in this study.

In addition to these analyses, the results are compared with the study of (Jabbari et al. [3]) for the same form of the FG hollow cylindrical object with the same values of material properties. The results are presented in Table 4. The compared results show that the proposed numerical solution procedure adequately provides seven-digit accuracy by using only nine collocation points.

Results of two different combinations with and thermal convection are examined graphically in Figure 1. The figure consists of four subfigures that show temperature, radial displacement, as well as radial and circumferential stresses called a, b, c and d respectively.

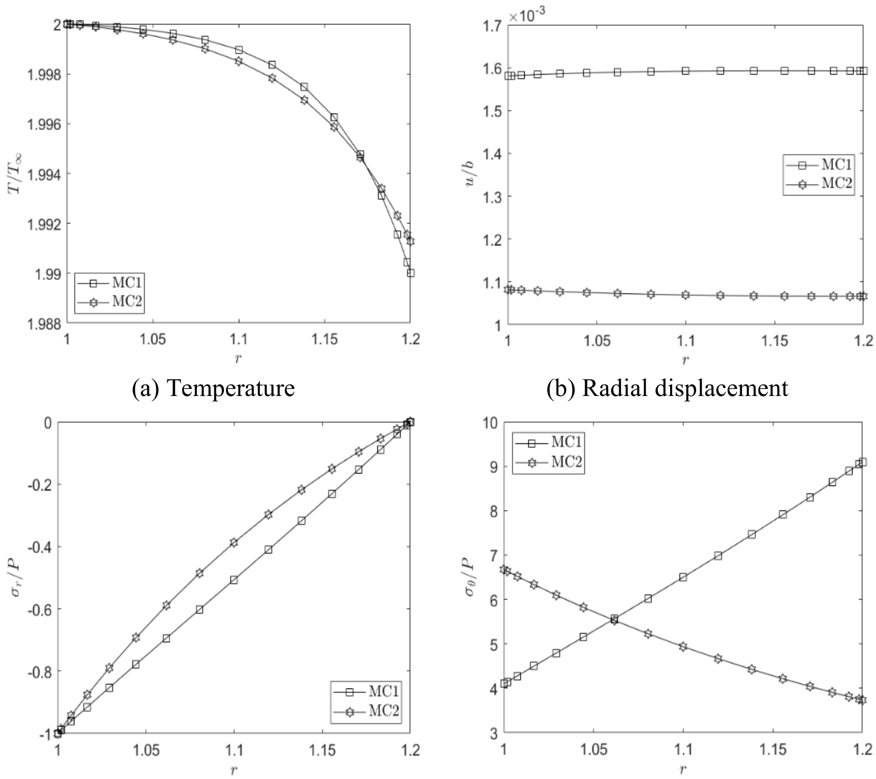
Table 3. Grid independence tests in the middle point of FG hollow cylinder wall (at r = 1.1 m)

	Cylinder	
Number of interval	$\boldsymbol{T/T_w}$	$\boldsymbol{u}$
2	0.9994105160	0.0019534844
4	0.9992357939	0.0019112216
6	0.9992524141	0.0019103517
8	0.9992525121	0.0019103411
10	0.9992525138	0.0019103410
12	0.9992525138	0.0019103410
14	0.9992525138	0.0019103410
16	0.9992525138	0.0019103410

Table 4. Comparison of Pseudospectral Chebyshev Method with the analytical study of Jabbari et al. [3] for FGM cylinder ($m = 2$)

	Temperature		Radial Displacement		Radial Stress		Circumferential Stress	
r	Analytic	Numeric	Analytic	Numeric	Analytic	Numeric	Analytic	Numeric
1.000000	1.00000000	1.00000000	0.00114082	0.00114082	–1.00000000	–1.00000000	4.51743419	4.51743418
1.007612	0.95073887	0.95073887	0.00113591	0.00113591	–0.95818342	–0.95818344	4.55316527	4.55316526
1.029289	0.81639345	0.81639345	0.00112256	0.00112256	–0.84103542	–0.84103542	4.65556819	4.65556818
1.061732	0.63049497	0.63049497	0.00110415	0.00110415	–0.67071728	–0.67071729	4.81063898	4.81063897
1.100000	0.43200601	0.43200601	0.00108458	0.00108458	–0.47680188	–0.47680188	4.99640328	4.99640327
1.138268	0.25319801	0.25319801	0.00106700	0.00106700	–0.28962709	–0.28962711	5.18532431	5.18532429
1.170711	0.11514260	0.11514260	0.00105344	0.00105344	–0.13565821	–0.13565821	5.34802579	5.34802578
1.192388	0.02911020	0.02911020	0.00104499	0.00104499	–0.03496771	–0.03496773	5.45807310	5.45807308
1.200000	–0.00000000	0.00000000	0.00104213	0.00104213	–0.00000000	0.00000000	5.49697463	5.49697462

The effect of two different material pairs, MC_1, MC_2 on a hollow FG cylinder is monitored in Figure 1. Due to the lower ambient temperature and convective boundary conditions, the temperature of the outer surface is lower compared to the inner surface (Fig. 1a). MC_1 reaches a lower temperature value than MC_2 at the outer radius in Figure 1a. Its clearly seen from Figure 1b that lower radial displacement occurs in MC_2. This is the result of choosing the inhomogeneity parameter as negative for the Young's modulus in the grading of MC_2. It is also due to the choice of Young's modulus closer to each other and relatively higher in grading material used in pairs.



(a) Temperature (b) Radial displacement

(c) Radial stress distribution (d) Circumferential stress distribution

Fig. 1. The dimensionless temperature, radial displacement, and radial and circumferential stress distributions of hollow FGM cylinder for materials MC_1 and MC_2

In Figure 1c, radial stress in the direction of compression and MC_2 has lower values than MC_1 in any point of wall thickness. On the outer radius, radial stress reduces to zero due to the traction free boundary condition. Although the circumferential stress is lower for MC_1 in the inner wall, it reverses on the outer wall, and MC_2 reaches lower stress value. Both MCs meet the same circumferential stress value around the wall midpoint. It can be concluded that MC_2 is relatively suitable due to keeping circumferential stress stable along the wall thickness.

5. Conclusions

In this study, thermal and mechanical stresses in a hollow thick-walled cylinder made of FGM under the effect of thermal convection coefficient is presented. All material properties are assumed to vary exponentially in the radial direction. These inhomogeneous material properties produce an irregular and variable coefficient two-point linear boundary value problem. This linear boundary value problem is solved numerically by the pseudospectral Chebyshev method. Benchmark solutions are used to verify the temperature, displacement and stress distributions in the form of tables. Results agree with the study of Jabbari et al. [3]. Furthermore, grid independence tests are conducted to emphasize the convergence of the numerical solutions. In the analysis, two different ceramic and metal mixtures are used as special materials. The effect of the two different mixtures and thermal convection on temperature, displacement and stresses are discussed extensively. In addition, the pseudospectral Chebyshev method used in this study, based on the definition of spatial fields using Chebyshev polynomials with grid density at the boundaries, is a method in which derivatives are calculated with high accuracy using very few collocation points. Therefore, this method has high accuracy, low calculation costs and ease of application and can be easily adapted to such engineering problems.

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