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THE THREE-BAND MATRICES

Grzegorz Biernat, Justyna Boryś, Ilona Całusińska, Agnieszka Surma

Institute of Mathematics, Czestochowa University of Technology, Poland
imi@imi.pcz.pl

1. Formulation of the problem

In practical problem with numeric description of heat flowing [1] (Fourier equation), the diffusion (Fick equation), etc. there are large linear system equations with 3-band matrices. In the paper we give the direct models for the determinants (property 1 and 4) and the algebraic complement of that matrices (property 2 and 5). In the result we receive the direct models for solving the linear systems equations expressed by the 3-band matrices (property 3 and 6).

2. Solution of the problem

Let

$$A_n = \begin{bmatrix} a & b & 0 & 0 & \dots & 0 \\ b & a & b & 0 & \dots & 0 \\ 0 & b & a & b & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & b & a & b \\ 0 & 0 & \dots & 0 & b & a \end{bmatrix}_{n\times n} \tag{1}$$

be the 3-band matrix of dimension n.

Property 1. The determinant of the matrix A_n is given by the formula

$$W_n = \det A_n = a^n - \binom{n-1}{1} a^{n-2} b^2 + \binom{n-2}{2} a^{n-4} b^4 - \binom{n-3}{3} a^{n-6} b^6 + \dots \tag{2}$$

Inductive proof. $W_1 = a.$ Suppose

$$W_k = \det A_k = a^k - \binom{k-1}{1} a^{k-2} b^2 + \binom{k-2}{2} a^{k-4} b^4 - \binom{k-3}{3} a^{k-6} b^6 + \dots \tag{3}$$

for $k = n-1$ and $k = n.$ Then

$$W_{n+1} = aW_n - b^2 W_{n-1} = a^{n+1} - \binom{n}{1} a^{n-1} b^2 + \binom{n-1}{2} a^{n-3} b^4 - \binom{n-2}{3} a^{n-3} b^4 + \ldots \quad (4)$$

This end the proof.

Property 2. The algebraic complement of the matrix A_n is given by the formula

$$A_{j\,j+l} = A_{j+l\,j} = (-1)^{2j+l} b^l W_{j-1} W_{n-j-l} \quad (5)$$

where $j \geq 1$ and $0 \leq l \leq n-j$.
Proof. A simple observation.

Property 3. Let

$$Z_n = \begin{array}{c} \begin{array}{ccccccc} & & & & \downarrow^{k} & & \end{array} \\ \begin{vmatrix} a & b & 0 & \ldots & c_1 & \ldots & 0 \\ b & a & b & \ldots & c_2 & \ldots & 0 \\ 0 & b & a & \ldots & c_3 & \ldots & 0 \\ \vdots & \vdots & \ddots & & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \ldots & c_{n-1} & a & b \\ 0 & 0 & \ldots & \ldots & c_n & b & a \end{vmatrix}_{n \times n} \end{array} \quad (6)$$

We have

$$Z_n = W_{n-k} \sum_{i=1}^{k-1} c_i (-1)^{i+k} b^{k-i} W_{i-1} + W_{k-1} \sum_{i=0}^{n-k} c_{k+i} (-1)^{2k+i} b^i W_{n-k-i}, \text{ where } W_0 = 1 \quad (7)$$

So, the linear system equations

$$\begin{bmatrix} a & b & 0 & 0 & \ldots & 0 \\ b & a & b & 0 & \ldots & 0 \\ 0 & b & a & b & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & b & a & b \\ 0 & 0 & \ldots & 0 & b & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_{n-1} \\ c_n \end{bmatrix} \quad (8)$$

gives

$$x_k = \frac{W_{n-k} \sum_{i=1}^{k-1} c_i (-1)^{i+k} b^{k-i} W_{i-1} + W_{k-1} \sum_{i=0}^{n-k} c_{k+i} (-1)^{2k+i} b^i W_{n-k-i}}{W_n} \quad (9)$$

for $1 \le k \le n$.

Let

$$B_n = \begin{bmatrix} a & b & 0 & 0 & \dots & 0 \\ b & a & b & 0 & \dots & 0 \\ 0 & b & a & b & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & b & a & b \\ 0 & 0 & \dots & 0 & 2b & a \end{bmatrix}_{n\times n} \tag{10}$$

be the quasi 3-band matrix of dimension n.

Property 4. The determinant of the matrix B_n is given by the formula

$$V_n = a^n - n\left(\binom{n-2}{0}a^{n-2}b^2 - \frac{1}{2}\binom{n-3}{1}a^{n-4}b^4 + \frac{1}{3}\binom{n-4}{2}a^{n-6}b^6 - \dots\right) \tag{11}$$

Proof. We have

$$V_n = W_n - b^2W_{n-2} = a^n - n\binom{n-2}{0}a^{n-2}b^2 + \frac{n}{2}\binom{n-3}{1}a^{n-4}b^4 - \frac{n}{3}\binom{n-4}{2}a^{n-6}b^6 + \dots \tag{12}$$

This end the proof.

Property 5. The algebraic complements of the matrix B_n are given by the formulas

$$B_{j\,j+l} = B_{j+l\,j} = b^lW_{j-1}V_{n-j-l}, \text{ where } j = 1,\dots,n-1 \text{ and } 0 \le l \le n-j-1 \tag{13}$$

$$B_{n\,j} = A_{n\,j} = b^{n-j}W_{j-1}, \qquad \text{where } j = 1,\dots,n \tag{14}$$

$$B_{j\,n} = 2A_{j\,n} = 2b^{n-j}W_{j-1}, \qquad \text{where } j = 1,\dots,n-1 \tag{15}$$

Proof. A simple observation.

Property 6. Let

$$U_n = \begin{array}{c} \hspace{5em}\downarrow^k \\ \begin{vmatrix} a & b & 0 & c_1 & \dots & 0 \\ b & a & b & c_2 & \dots & 0 \\ 0 & b & a & c_3 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c_{n-1} & a & b \\ 0 & 0 & \dots & c_n & 2b & a \end{vmatrix}_{n\times n} \end{array} \tag{16}$$

i) For $k = 1,2,\dots,n-1$ we have

$$U_n = V_{n-k}\sum_{i=1}^{k-1} c_i(-1)^{i+k} b^{k-i} W_{i-1} + W_{k-1}\sum_{i=0}^{n-k} c_{k+i}(-1)^{2k+i} b^i V_{n-k-i},\ \text{where } V_0 = 1 \quad (17)$$

ii) and for $k = n$ we have

$$\begin{array}{c} \downarrow^{k=n} \\ \begin{vmatrix} a & b & 0 & 0 & \dots & c_1 \\ b & a & b & 0 & \dots & c_2 \\ 0 & b & a & b & \dots & c_3 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & b & a & c_{n-1} \\ 0 & 0 & \dots & 0 & 2b & c_n \end{vmatrix}_{n\times n} \end{array} = 2\sum_{i=1}^{n-1} c_i(-1)^{i+n} b^{n-i} W_{i-1} + c_n W_{n-1} \quad (18)$$

So, the linear system equations

$$\begin{bmatrix} a & b & 0 & 0 & \dots & 0 \\ b & a & b & 0 & \dots & 0 \\ 0 & b & a & b & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & b & a & b \\ 0 & 0 & \dots & 0 & 2b & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_{n-1} \\ c_n \end{bmatrix} \quad (19)$$

gives

$$x_k = \frac{V_{n-k}\sum_{i=1}^{k-1} c_i(-1)^{i+k} b^{k-i} W_{i-1} + W_{k-1}\sum_{i=0}^{n-k} c_{k+i}(-1)^{2k+i} b^i V_{n-k-i}}{V_n} \quad (20)$$

for $k = 1, 2, \dots, n-1$ and

$$x_n = \frac{2\sum_{i=1}^{n-1} c_i(-1)^{i+n} b^{n-i} W_{i-1} + c_n W_{n-1}}{V_n} \quad (21)$$

for $k = n$.

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