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# ON MELLIN TRANSFORM APPLICATION TO SOLUTION OF FRACTIONAL DIFFERENTIAL EQUATIONS

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**Abstract.** The Mellin transform method is applied to fractional differential equations with a right-sided derivative and variable potential. After solving the intermediate difference equation we arrive at the general solution of the problem in the form of a Meijer-G-function series. Using the symmetry properties of fractional derivatives we transform it into a general solution for an analogous equation with the left-sided Riemann-Liouville derivative. Two examples are studied in detail and an explicit form of component solutions is given.

## Introduction

Fractional calculus, involving derivatives and integrals of non-integer order, is now an integral part of mathematical modelling methods. Since its success in description of anomalous diffusion [1-3], non-integer order calculus, both in one and multidimensional space, it has become an important tool in many areas of physics, mechanics, chemistry, engineering, finances and bioengineering (see monographs [4-7] and the references therein).

In applications of fractional calculus, a new class of integral-differential equations has been developed. They include integrals and derivatives of non-integer order and in general variable coefficients. Solutions have been studied for two decades [8-10] and the methods of solving include fixed point theorems, integral transform methods as well as operational methods based on properties of new classes of special functions. Still, some fractional differential equations, including Euler-Lagrange equations of fractional mechanics, remain an open problem.

In this paper we test the Mellin transform method for fractional differential equations with the right-sided derivative and variable potential - $t^{\beta}$. The main motivation to study this class of equations is the fact that the solutions are known in literature only for its analogue with the left-sided derivative [10]. Kilbas and Saigo solved such an equation using the Banach theorem on a fixed point, Klimek and Dziembowski in [11] proposed to apply the Mellin transform. This method seems more suitable for equations with right-sided operators as in this case, fixed point theorems yield only the existence and uniqueness of solutions. Attempts to calculate the exact form of solutions by means of straightforward integration did not produce an explicit and closed form of solution.

The application of the Mellin transform was also earlier proposed in solving procedure for some equations with a fractional differential operator of a variational type [12, 13]. We shall obtain the solutions in an analogous form in the present article. Using the symmetry properties of fractional derivatives with respect to reflection in a finite time interval we shall solve an equation with the left-sided derivative and potential $(b\text{-}t)^{\beta}$. Two examples are studied in Sections 3.1 and 3.2. In the case of constant potential $\beta = 0$ the discussed equation becomes an eigenfunction equation for the right-sided derivative and its solution is a linear combination of generalized Mittag-Lefller functions [10]. In the second example we assume $\beta = -\alpha/2$ and using the reduction property for Meijer G-functions we obtain a simple form of solution.

The paper is organized as follows: Section 1 contains necessary and relevant definitions and formulas from fractional calculus, in Section 2 we include the definition and properties of the Mellin transform. The main results are enclosed in Section 3 - in Propositions 3.1 and 3.3. The procedure of solving the equation is explained in detail and two examples of application are discussed.

## 1. Fractional integrals and derivatives

We shall recall the definitions of right-sided fractional operators [10, 14]. The following formulas describe fractional integrals for order $\alpha$ fulfilling condition $\mathrm{Re}(\alpha) > 0$:

$$\left(I^{\alpha}_{b-} f\right)(t) := \frac{1}{\Gamma(\alpha)} \int_t^b \frac{f(u)\, du}{(u-t)^{1-\alpha}} \quad 0 < t < b \tag{1}$$

$$\left(I^{\alpha}_{-} f\right)(t) := \frac{1}{\Gamma(\alpha)} \int_t^\infty \frac{f(u)\, du}{(u-t)^{1-\alpha}} \tag{2}$$

The first of the above integrals is defined in a finite time interval and it is called a Riemann-Liouville right-sided integral. The second formula describes a right-sided integral defined on the real halfaxis, namely it is a Liouville integral.
Using the above fractional integrals we define the following fractional derivatives

$$\left(D^{\alpha}_{b-} f\right)(t) := \left(-\frac{d}{dt}\right)^n \left(I^{n-\alpha}_{b-} f\right)(t) \quad 0 < t < b \tag{3}$$

$$\left(D^{\alpha}_{-} f\right)(t) := \left(-\frac{d}{dt}\right)^n \left(I^{n-\alpha}_{-} f\right)(t) \tag{4}$$

where $\mathrm{Re}(\alpha) > 0$ and $[\mathrm{Re}(\alpha)] = n$-1.
Operator $D^{\alpha}_{b-}$ is correctly determined in a finite time interval and it is known as a right-sided Riemann-Liouville derivative. Derivative $D^{\alpha}_{-}$ acts on functions defined on the real halfaxis and it is called a right-sided Liouville derivative.

The following formulas present some results of fractional integration and differentiation for power functions:

$$I^{\alpha}_{b-}(b-t)^{\beta-1} = \frac{\Gamma(1-\alpha-\beta)}{\Gamma(1-\beta)}(b-t)^{\beta+\alpha-1} \quad \mathrm{Re}(\beta) > 0 \tag{5}$$

$$I^{\alpha}_{-}t^{\beta-1} = \frac{\Gamma(1-\alpha-\beta)}{\Gamma(1-\beta)}t^{\beta+\alpha-1} \quad \mathrm{Re}(\alpha+\beta) < 1 \tag{6}$$

$$D^{\alpha}_{b-}(b-t)^{\beta-1} = \frac{\Gamma(1+\alpha-\beta)}{\Gamma(1-\beta)}(b-t)^{\beta-\alpha-1} \quad \mathrm{Re}(\beta) > 0 \tag{7}$$

$$D^{\alpha}_{-}t^{\beta-1} = \frac{\Gamma(1+\alpha-\beta)}{\Gamma(1-\beta)}t^{\beta-\alpha-1} \quad \mathrm{Re}(\alpha+\beta) - [\mathrm{Re}\alpha] < 1 \tag{8}$$

In what follows we also apply the composition rule for fractional derivatives and integrals of the same order. When $\mathrm{Re}(\alpha) > 0$ and $f \in L_1(0,b)$, the following formula

$$\left(D^{\alpha}_{b-} I^{\alpha}_{b-} f\right)(t) = f(t) \tag{9}$$

is valid almost everywhere in interval $[0,b]$. In the case when $\mathrm{Re}(\alpha) > 1$ the above composition rule holds at any point $t \in [0,b]$.

## 2. Mellin transform and its properties

Let us recall the definition of the Mellin transform (here $s$ is a complex variable and compare monograph [15]):

$$\mathrm{M}\left[f\right](s) := \int_0^{\infty} t^{s-1} f(t)dt \tag{10}$$

Similarly to the Laplace and Fourier integral transforms the Mellin transform also has its convolution defined by formula

$$f(t) * g(t) := \int_0^{\infty} f(u) g\left(\frac{t}{u}\right) \frac{du}{u} \tag{11}$$

Acting on the Mellin convolution of two functions, the Mellin transform yields the product of transforms of respective functions:

$$\mathrm{M}\,[f*g](s)=\mathrm{M}\,[f](s)\cdot\mathrm{M}\,[g](s) \tag{12}$$

The following two properties shall be applied to solving fractional differential equation (15). The first formula describes shifting property

$$\mathrm{M}\,[t^{\gamma}f(t)](s)=\mathrm{M}\,[f(t)](s+\gamma) \tag{13}$$

The Mellin transform is known for left- and right-sided fractional operators acting on function *f*. Here we quote the formula for the right-sided Liouville integral, which we shall apply in our calculations in Section 3 [10]:

$$\mathrm{M}\,[I_{-}^{\alpha}f](s)=\frac{\Gamma(s)}{\Gamma(s+\alpha)}\mathrm{M}\,[f](s+\alpha) \qquad Re(s)>0 \tag{14}$$

## 3. Fractional differential equation with right-sided Riemann-Liouville derivative and variable potential

Let us consider the following fractional differential equation in a finite time interval

$$\left(D_{b-}^{\alpha}-\lambda t^{\beta}\right)f(t)=0, \qquad t\in[0,b] \tag{15}$$

We shall extend this equation to the real positive halfaxis. To this aim we rewrite solution *f* in the following form

$$f_0(t):=f(t)\Delta H(t) \qquad t\in R_{+}$$

where we denote $\Delta H(t)$: = $H(t)-H(t-b)$ with $H$ - being the Heaviside's function.
We notice that function $f_0$ vanishes outside interval [0,b] and coincides with solution *f* inside time interval [0,b]. Hence we replace equation (15) with an equation in the form of

$$\left(D_{b-}^{\alpha}-\lambda t^{\beta}\right)f_0(t)=0 \tag{16}$$

The above equation can be transformed to its equivalent integral form using composition rule (9). Thus, we shall obtain function $f_0$, solving integral equation of order $\alpha$:

$$\left(1-\lambda I_{b-}^{\alpha}t^{\beta}\right)f_0(t)=f_{\alpha}^{st}(t) \tag{17}$$

where function $f_\alpha^{st}$ is an arbitrary stationary function of the right-sided Riemann--Liouville derivative, given by formula

$$f_\alpha^{st}(t) = \sum_{l=1}^{n} c_k (b-t)^{\alpha-k} \Delta H(t) \tag{18}$$

Let us note that the following two fractional integrals coincide on positive halfaxis for solution $f_0$

$$I_{b-}^{\alpha} f_0(t) \equiv I_{-}^{\alpha} f_0(t)$$

and owing to this property we can rewrite equation (17) as an equation with a Liouville fractional integral

$$\left(1 - \lambda I_{-}^{\alpha} t^{\beta}\right) f_0(t) = f_\alpha^{st}(t) \tag{19}$$

The next step is the application of the Mellin transform. The result is a difference equation for the Mellin transform of solution $\mathsf{M}\left[f_0\right]$:

$$\left(1 - \lambda g(s) T_{\alpha+\beta}\right) \mathsf{M}\left[f_0\right](s) = \mathsf{M}\left[f_\alpha^{st}\right](s) \qquad \mathrm{Re}(s) > 0 \tag{20}$$

where $T_{\alpha+\beta}$ is a translation operator acting on functions as follows: $T_{\alpha+\beta}\, h(s) =$ $= h(s+\alpha+\beta)$ and we assume $\mathrm{Re}(\alpha+\beta) > 0$. Function $g$ is given as the quotient of Euler Gamma functions:

$$g(s) = \frac{\Gamma(s)}{\Gamma(s+\alpha)}$$

The above difference equation is solved by the following series absolutely convergent in complex halfplane $\mathrm{Re}(s) > 0$ when $\mathrm{Re}(\alpha+\beta) > 0$:

$$\mathsf{M}\left[f_0\right](s) = \sum_{m=0}^{\infty} \lambda^m \left(g(s) T_{\alpha+\beta}\right)^m \mathsf{M}\left[f_\alpha^{st}\right](s) \tag{21}$$

In order to derive an exact and explicit form of solution $f_0$ we should now invert the above Mellin transform. Let us note that for each $m$ the last term is simply the Mellin transform of the stationary function given in formula (18). The main difficulty is therefore the inverse Mellin transform for the corresponding products of the $g$-functions. However, they can be identified as kernels $\mathcal{G}$ of the Mellin transforms defining the Meijer G-functions (compare formula (1.1.1) from the monograph by Kilbas and Saigo [10]) for each natural number $m$:

$$\prod_{l=0}^{m-1} g(s+l(\alpha+\beta)) = \mathrm{G}_{m,m}^{m,0}\left[\begin{array}{c}\vec{a}_m\\ \vec{b}_m\end{array}\middle| s\right] \tag{22}$$

where the vectors are defined as follows

$$\vec{a}_m := (\alpha+\beta)\vec{j}_m + \alpha\,\vec{e}_m \qquad \vec{b}_m := (\alpha+\beta)\vec{j}_m \tag{23}$$

$$\vec{j}_m := [0, 1, \ldots, m-1] \in \boldsymbol{R}^m \qquad \vec{e}_m := [1, \ldots, 1] \in \boldsymbol{R}^m$$

Let us note that formula (21), describing the Mellin transform of solution $f_0$, holds in complex halfplane Re($s$) > 0. Hence, inverting this transform we should use vertical contour $L_{i\gamma\infty}$. According to Theorem 1.1 and formulas (1.1.7-1.1.10) from monograph [16] we calculate parameters

$$\Delta = 0 \qquad a^* = 0 \qquad \mu = -m\alpha$$

and obtain the following condition

$$\Delta\gamma + \mathrm{Re}(\mu) < -1 \quad \Rightarrow \quad \mathrm{Re}(\alpha) > \frac{1}{2}$$

Applying formula (12) for convolution and formula (22) we obtain the solution given as the series of Meijer G-functions in the convolution with the stationary function multiplied by a power function

$$f_0(t) = \sum_{m=0}^{\infty} \lambda^m G_{m,m}^{m,0}\left[\begin{array}{c}\vec{a}_m\\ \vec{b}_m\end{array}\middle| t\right] * t^{m(\alpha-\beta)} f_\alpha^{st}(t) \tag{24}$$

In order to calculate the Mellin convolutions we apply the integration formula for Fox H-functions rewritten for a subset of Meijer G-functions (compare Theorem 2.7 from the monograph by Kilbas and Saigo [16]). When Δ = 0 and condition

$$\max_{1\le i\le n}\left[\mathrm{Re}(a_i)-1\right] + \mathrm{Re}(\omega) + \alpha - k + 1 < 0 \tag{25}$$

is fulfilled, then the following general formula for the integration of Meijer G-functions is valid

$$I_-^{\alpha-k+1} u^\omega G_{p,q}^{m,n}\left[\begin{array}{c}\vec{a}\\ \vec{b}\end{array}\middle| u\right](x) =$$

$$= x^{\omega+\alpha-k+1} G_{p+1,q+1}^{m+1,n}\left[\begin{array}{c}[\vec{a}, -\omega]\\ \left[-\omega-\alpha+k-1, \vec{b}\right]\end{array}\middle| x\right]$$

As assumption (25) is fulfilled for each element of series (24), we apply the above formula for integration and we calculate the explicit form of all Mellin convolutions in series (24)

$$G^{m,0}_{m,m}\left[\begin{array}{c|c}\vec{a}_m & t\\ \vec{b}_m & \end{array}\right]*t^{m(\alpha-\beta)}(b-t)^{\alpha-k}\Delta H(t)=$$

$$=t^{m(\alpha+\beta)}b^{\alpha-k}\Gamma(\alpha-k+1)I_{-}^{\alpha-k+1}\,u^{-m(\alpha+\beta)-\alpha+k-1}G^{m,0}_{m,m}\left[\begin{array}{c|c}\vec{a}_m & u\\ \vec{b}_m & \end{array}\right]\left(\frac{t}{b}\right)=$$

$$=b^{m(\alpha+\beta)+\alpha-k}\Gamma(\alpha-k+1)G^{m+1,0}_{m+1,m+1}\left[\begin{array}{c|c}\vec{A}_{m,k} & \frac{t}{b}\\ \vec{B}_{m,k} & \end{array}\right]$$

where the vectors defining the Meijer G-functions look as follows

$$\vec{A}_{m,k}:=[\vec{a}_m, m(\alpha+\beta)+\alpha-k+1]\in \boldsymbol{R}^{m+1} \tag{26}$$

$$\vec{B}_{m,k}:=[m(\alpha+\beta),\vec{b}_m]\in \boldsymbol{R}^{m+1} \tag{27}$$

The above integration formula yields the Mellin convolution corresponding to the $(b\text{-}t)^{\alpha\text{-}k}$ component of the stationary function. Using this formula we arrive at the component series of the solution connected to the respective component of the stationary function:

$$f_0^k(t)=(b-t)^{\alpha-k}+b^{\alpha-k}\Gamma(\alpha-k+1)\sum_{m=1}^{\infty}\left(\lambda b^{\alpha+\beta}\right)^m G^{m+1,0}_{m+1,m+1}\left[\begin{array}{c|c}\vec{A}_{m,k} & \frac{t}{b}\\ \vec{B}_{m,k} & \end{array}\right]$$

The general solution of equations (19) and (15) is an arbitrary linear combination of the above series

$$f_0(t)=\sum_{k=1}^{n}c_k f_0^k(t) \qquad t\in[0,b]$$

The presented considerations can be summarized in the following Proposition.

**Proposition 3.1**
*Let* $\mathrm{Re}(\alpha)\in(n-1,n)$ *and* $\mathrm{Re}(\alpha+\beta)>0$, $\mathrm{Re}(\alpha)>1/2$. *Then equation*

$$\left(D^{\alpha}_{b-}-\lambda t^{\beta}\right)f(t)=0$$

*has in finite time interval [0,b] a general solution in the form of a linear combination of solutions* $f_0^k$

$$f_0(t) = \sum_{k=1}^{n} c_k f_0^k(t) \tag{28}$$

*where* $c_k$ *are arbitrary real coefficients and components* $f_0^k$ *are the following series of Meijer G-functions*

$$\begin{aligned} f_0^k(t) &= (b-t)^{\alpha-k} + \\ &+ b^{\alpha-k}\Gamma(\alpha-k+1)\sum_{m=1}^{\infty}\left(\lambda b^{\alpha+\beta}\right)^m G_{m+1,m+1}^{m+1,0}\left[\begin{array}{c|c} \vec{A}_{m,k} & \frac{t}{b} \\ \vec{B}_{m,k} & \end{array}\right] \end{aligned} \tag{29}$$

*with defining vectors given by formulas (26, 27).*

**Remark 3.2**

Let us note that the Riemann-Liouville derivatives obey in a finite time interval the following formula [14]:

$$Q\,D_{b-}^{\alpha} = D_{0+}^{\alpha}\,Q \tag{30}$$

where operator $Q$ is a reflection operator in interval $[0,b]$ acting as follows on functions determined in this interval $Qh(t) := h(b-t)$ and $D_{0+}^{\alpha}$ is the left-sided Riemann-Liouville derivative [10, 14].

Let us observe that this remark yields the following Proposition valid.

**Proposition 3.3**

*Let* $\mathrm{Re}(\alpha) \in (n-1, n)$ *and* $\mathrm{Re}(\alpha+\beta) > 0$, $\mathrm{Re}(\alpha) > 1/2$. *Then equation*

$$\left(D_{0+}^{\alpha} - \lambda(b-t)^{\beta}\right) f(t) = 0$$

*has in finite time interval* $[0,b]$ *a general solution in the form of a linear combination of solutions* $f_0^k$

$$f_0(t) = \sum_{k=1}^{n} c_k f_0^k(t) \tag{31}$$

*where* $c_k$ *are arbitrary real coefficients and components* $f_0^k$ *are the following series of Meijer G-functions*

$$f_0^k(t) = t^{\alpha-k} + \\ + b^{\alpha-k}\Gamma(\alpha-k+1)\sum_{m=1}^{\infty}\left(\lambda b^{\alpha+\beta}\right)^m G_{m+1,m+1}^{m+1,0}\left[\begin{array}{c}\vec{A}_{m,k}\\ \vec{B}_{m,k}\end{array}\middle| 1-\frac{t}{b}\right] \tag{32}$$

*with defining vectors given by formulas (26, 27).*

### 3.1. Example: solution of equation (15) for $\alpha+\beta=\alpha$

Let us consider an example for $\beta = 0$. Equation (15) becomes an eigenvalue equation with complex eigenvalue λ:

$$\left(D_{b-}^{\alpha} - \lambda\right) f_0(t) = 0 \qquad t \in [0, b]$$

The Meijer G-functions appearing in formulas (28, 29) for the general solution can be transformed using their reduction property (see Property 2.2 in monograph [16]). We conclude that in this case they can be identified with the power functions:

$$G_{m+1,m+1}^{m+1,0}\left[\begin{array}{c}\vec{A}_{m,k}\\ \vec{B}_{m,k}\end{array}\middle|\frac{t}{b}\right] = G_{1,1}^{1,0}\left[\begin{array}{c}[m\alpha+\alpha-k+1]\\ [0]\end{array}\middle|\frac{t}{b}\right] = \\ = \frac{\Gamma(\alpha-k+1)}{\Gamma(m\alpha+\alpha-k+1)}\left(1-\frac{t}{b}\right)^{m\alpha+\alpha-k}$$

Hence component solutions $f_0^k$ for $k = 1,\ldots,n$ are of the form

$$f_0^k(t) = (b-t)^{\alpha-k} E_{\alpha,\alpha-k+1}(\lambda(b-t)^{\alpha})$$

where functions $E$ are the generalized Mittag-Lefller functions (see their definition and properties in monograph [10]).

### 3.2. Example: solution of equation (15) for $\alpha+\beta=\alpha/2$

As a next example we solve an equation for $\beta = -\alpha/2$. For this value of parameter $\beta$, equation (15) has the following form:

$$\left(D_{b-}^{\alpha} - \lambda t^{-\alpha/2}\right) f_0(t) = 0 \qquad t \in [0, b]$$

Similarly to the previous example we can also apply the reduction formula and obtain for $m > 0$ the following simple formulas for Meijer G-functions

$$G^{m+1,0}_{m+1,m+1}\left[\begin{matrix}\vec{A}_{m,k}\\ \vec{B}_{m,k}\end{matrix}\left|\frac{t}{b}\right.\right]=G^{2,0}_{2,2}\left[\begin{matrix}[\alpha/2(m+1),\alpha/2(m+2)-k+1]\\ [0,\alpha/2]\end{matrix}\left|\frac{t}{b}\right.\right]$$

Component solution $f_0^k$ for $k=1,\ldots,n$ looks as follows

$$f_0^k(t)=(b-t)^{\alpha-k}+$$

$$+b^{\alpha-k}\Gamma(\alpha-k+1)\sum_{m=1}^{\infty}\left(\lambda b^{\alpha/2}\right)^m G^{2,0}_{2,2}\left[\begin{matrix}[\alpha/2(m+1),\alpha/2(m+2)-k+1]\\ [0,\alpha/2]\end{matrix}\left|\frac{t}{b}\right.\right]$$

## Conclusions

In the paper we derived the general solution of two fractional differential equations with variable potential. The developed method of solving includes:

- Mellin transform,
- solution of intermediate difference equation,
- inversion of Mellin transform with application of Meijer G-functions theory,
- application of reflection operator in the finite time interval.

Using the above procedure we obtained the general solution of the equation with the right-sided Riemann-Liouville derivative in the form of a Meijer G-function series. Afterwards, by means of the reflection operator, this series also yields general solution of an equation with a left-sided fractional derivative.
The obtained solutions will be applied in a subsequent paper in solving the generalized fractional equations in the form of

$$L\,[t^{-\beta}D^{\alpha}_{b-}\,]\,f(t)=0$$

$$L\,[(b-t)^{-\beta}D^{\alpha}_{0+}\,]\,f(t)=0$$

where $L$ is an arbitrary polynomial function.

Concluding, the Mellin transform method seems to be a more general tool than the Laplace and Fourier transforms widely applied in fractional differential equations. Although it yields solutions in a complicated form and their properties are still to be investigated, the Mellin transform allows us to solve equations with operators of a variational type [17]-[19].

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